TRIGONOMETRI II

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Oleh :

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P A D A N G

1991

KATA PENGANTAR

Berkat rahmat Tuhan Yang Maha Esa dan sesuai dengan kemampuan yang ada, buku dengan judul **"Trigonometri II"** telah dapat disusun sebagaimana mestinya.

Buku Trigonometri II ini, merupakan lanjutan dari buku Trigonometri I. Buku ini penulis susun guna melengkapi bahan bacaan para pembaca yang berminat terhadap mata kuliah Trigonometri II khususnya dan bidang studi Matematika umumnya.

Penulis menyadari bahwa buku ini mungkin ada kekurangan-kekurangan. Oleh sebab itu kritik yang sehat dan membangun dari para pembaca diterima dengan senang hati.

Akhirnya penulis mengucapkan terima kasih pada Jurusan Pendidikan Matematika FPMIPA IKIP Padang, yang telah bersedia membantu dalam pengetikan buku Trigonometri II ini.

Padang, Januari 1991

Penulis.

DAFTAR ISI

PERSAMAAN TRIGONOMETRI DARI JUMLAH 2 BUAH SUDUT

(lanjutan)

Uraian:

Menurut P.Wijdenes hal.112 persamaan bentuk:

1. $p \sin(x + a) = q \sin(x + b)$ dapat diselesaikan dengan rumus sinus.

   Penyelesaiannya adalah sebagai berikut:

   $p \sin(x + a) = q \sin(x + b)$

   $p(\sin x \cos a + \cos x \sin a) = q(\sin x \cos b + \cos x \sin b)$

   $p(\sin x \cos x + p \cos x \sin a = q \sin x \cos b + q \cos x \sin b$

   $(p \cos a - q \cos b)\sin x = (q \sin b - p \sin a) \cos x$

   $$\operatorname{tg} x = \frac{q \sin b - p \sin a}{p \cos a - q \cos b}$$

   $$= - \frac{p \sin a - q \sin b}{p \cos a - q \cos b}$$

   jika a, b, p dan q diketahui, maka harga x dapat dicari, untuk jelasnya perhatikan contoh 1 : pada hal.4.

   Persamaan bentuk $p \cos(x + a) = q(x + b)$ dan

   $p \operatorname{tg}(x + a) = q \operatorname{tg}(x + b)$

   juga dapat diselesaikan menurut prinsip-prinsip penyelesaian di atas.

2. **Persamaan kuadrat perbandingan trigonometri.**

   Persamaan kuadrat perbandingan trigonometri, suku-sukunya terdiri dari fungsi trigonometri. Menurut Wijdenes (1953), persamaan ini berasal dari bentuk-bentuk:

   a). $a \cos 2x + b \sin x + c = 0$

   b). $a \cos 2x + b \cos x + c = 0$

   c). $a \operatorname{tg} x + b \operatorname{cotg} x + c = 0$.

   Dengan mempergunakan rumus sudut rangkap, maka persamaan a) berubah menjadi bentuk:

   $A \sin^2 x + B \sin x + C = 0$, persamaan b) berubah menjadi bentuk: $A \cos^2 x + B \cos x + C = 0$, dan persamaan c) dapat dirubah menjadi bentuk: $A \operatorname{tg}^2 x + B \operatorname{tg} x + C = 0$.

Cara penyelesaian selanjutnya adalah dengan mempergunakan rumus abc atau pemfajtoran menurut aljabar.
Harga $B^2 - 4\ AC \geq 0$, dan perlu diingat bahwa $-1 \leq \sin x \leq 1$ dan $-1 \leq \cos x \leq 1$ untuk sembarang nilai x.
Untuk jelasnya dapat dilihat pada contoh 2 dan 3 pada halaman 4 dan 5.

3. Persamaan-persamaan yang dapat diselesaikan dengan bentuk $a \cos x + b \sin x = c$.

a). $a \operatorname{tg} x + b \operatorname{cotg} x + c = 0$.

Menurut Pwydenes (1953. hal.120) bentuk persamaan:

$$a \operatorname{tg} x + b \operatorname{cotg} x + c = 0$$

disamping dapat diselesaikan dengan membawa ke persamaan kuadrat, juga dapat diselesaikan kebentuk:

$$a \cos x + b \sin x = c$$

caranya adalah sebagai berikut:

$$a \operatorname{tg} x + b \operatorname{cotg} x + c = 0 (a \neq 0\ ,\ b \neq 0)$$

$$\frac{a \sin x}{\cos x} + \frac{b \cos x}{\sin x} + c = 0$$

$$\frac{a \sin^2 x + b \cos^2 x}{\sin x \cos x} = -c$$

$$a \sin^2 x + b \cos^2 x = -c \sin x \cos x$$

$$\frac{a(1 - \cos 2x)}{2} + \frac{b(1 + \cos 2x)}{2} = -c \sin x \cos x$$

$$a - a \cos 2x + b + b \cos 2x = -c \sin 2x$$

$$(-a + b)\cos 2x + c \sin 2x = -(a + b)$$

$$(a - b)\cos 2x - c \sin 2x = a + b$$

jika a, b dan c diketahui, maka harga x dapat dicari.

b). $\sin x \sin(x - a) = p \cos^2 x$

$$2 \sin x \sin(x - a) = 2p \cos^2 x$$

$$\cos a - \cos(2x - a) = p(1 + \cos 2x)$$

$$\cos a - \cos 2x \cos a - \sin 2x \sin a = p + p \cos 2x$$

$$(p + \cos a)\cos 2x + \sin a \sin 2x = \cos a - p$$

atau

$$A \cos 2x + B \sin 2x = C$$

persamaan dapat diselesaikan dengan syarat:

$$A^2 + B^2 \geq C^2$$

atau

$$(p + \cos a)^2 + \sin^2 a \geq (\cos a - p)^2$$
$$\sin^2 a + 4p \cos a \geq 0.$$

c). $a \sin^2 x + b \sin x \cos x + c \cos^2 x = d$

$2a \sin^2 x + 2b \sin x \cos x + 2c \cos^2 x = 2d$

$a(1 - \cos 2x) + b \sin 2x + c(1 + \cos 2x) = 2d$

$(c - a)\cos 2x + b \sin 2x = 2d - a - c$

dan seterusnya.

Persamaan ini dapat diselesaikan dengan syarat:

$$(c - a)^2 + b^2 \geq (2d - a - c)^2$$

Untuk menyelesaikan soal persamaan trigonometri bentuk

$$a \sin x \cos x + b(\cos x \pm \sin x) + c = 0$$

terlebih dahulu diumpamakan:

$$\cos x \pm \sin x = y$$
$$\cos^2 x + 2 \sin x \cos x + \sin^2 x = y^2$$
$$2 \sin x \cos x = y^2 - 1$$

sehingga bentuk persamaan di atas menjadi:

$$ay^2 - 2by - (2c + a) = 0.$$

Jika a, b, c dan d tertentu, maka penyelesaian selanjutnya dapat dicari.

<u>Contoh-contoh</u>:

1. Hitunglah x dari persamaan:

   $17 \sin(x + 25^o) = 15 \sin(x + 36^o)$.

   Penyelesaiannya:

   $17 \sin(x + 25^o) = 15 \sin(x + 36^o)$

   $17(\sin x \cos 25 + \cos x \sin 25^o) = 15(\sin x \cos 36 + \cos x \sin 36^o)$

$$17 \sin x \cos 25 + 17 \cos x \sin 25 = 15 \sin x \cos 36 + 15 \cos x \sin 36$$

$$\sin x(17 \cos 25 - 15 \cos 36) = \cos x(15 \sin 36 - 17 \sin 25)$$

$$\frac{\sin x}{\cos x} = \frac{15 \sin 36^\circ - 17 \sin 25^\circ}{17 \cos 25^\circ - 15 \cos 36^\circ}$$

$$tg\ x = \frac{15 \cdot 0{,}58779 - 17 \cdot 0{,}42262}{17 \cdot 0{,}90631 - 15 \cdot 0{,}80902}$$

$$= \frac{8{,}81685 - 7{,}18454}{15{,}40727 - 12{,}1353}$$

$$= \frac{1{,}63231}{3{,}27197} = 0{,}49888 \longrightarrow$$

$$x = 26^\circ 30' 50'' + k.180^\circ$$

(dengan menggunakan tabel logaritma).

2. Hitunglah x dari persamaan $3 + \cos 2x = 8 \cos x$.

Penyelesaian:

$3 + \cos 2x = 8 \cos x$

$3 + 2 \cos^2 x - 1 = 8 \cos x$

$2 \cos^2 x - 8 \cos x + 2 = 0$

$\cos^2 x - 4 \cos x + 1 = 0$

$$(\cos x)_{1.2} = \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$(\cos x)_1 = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} \quad \text{(tidak memenuhi)}$$

$$(\cos x)_2 = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3} = 0{,}26790$$

dengan mempergunakan daftar logaritma hal.89, untuk $\cos x = 0{,}26790$, maka:

$x_1 = 74^\circ 27' 39'' + k.360^\circ$

$x_2 = 285^\circ 32' 21'' + k.360^\circ$.

3. Hitunglah x dari persamaan: $tg\ x + 4\ cotg\ x = 5$.

Penyelesaian:

$$tg\ x + 4\ cotg\ x = 5$$

$$tg\ x + \frac{4}{tg\ x} = 5$$

$$tg^2x + 4 = 5\ tg\ x$$

$$tg^2x - 5\ tg\ x + 4 = 0$$

$$(tg\ x)_{1.2} = \frac{5 \pm \sqrt{25 - 16}}{2}$$

$$= \frac{5 \pm 3}{2}$$

dengan mempergunakan daftar logaritma hal.88, untuk

$tg\ x = \frac{5 + 3}{2} = 4$, maka:

harga $x = 75^o57'49" + k.180^o$

dan untuk $tg\ x = \frac{5 - 3}{2} = 1$

hagra $x = 45^o + k.180^o$.

4. Tentukanlah x dari persamaan $2\cos x + \cos 2x = 1$ dalam interval $0^o \leq x \leq 360^o$.

Penyelesaian:

$$2\cos x + \cos 2x = 1$$

$$2\cos x + 2\cos^2x - 1 = 1$$

$$2\cos^2x + 2\cos x - 2 = 0$$

$$\cos^2x + \cos x - 1 = 0$$

$$\cos x = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{-1 + \sqrt{5}}{2} = 0,61803$$

$x_1 = 51^o49'38" + k.360^o$

$x_2 = 308^o10'22" + k.360^o$

$\cos x = \frac{-1 - \sqrt{5}}{2} = -1,081$ (tidak memenuhi).

5. Hitunglah x dari persamaan:

$89425\cos x + 25616\sin x = 92800$.

$$89425(\cos x + \frac{25616}{89425}\sin x) = 92800$$

umpama $\operatorname{tg} \varphi = \dfrac{25616}{89425}$

$$\varphi = 15^o 59'4''$$

$$\cos x + \operatorname{tg}\varphi \sin x = \frac{92800}{89425}$$

$$\cos x + \frac{\sin \varphi}{\cos \varphi}\sin x = \frac{92800}{89425}$$

$$\cos x \cos \varphi + \sin x \sin \varphi = \frac{92800}{89425}\cos \varphi$$

$$\cos(x - \varphi) = \frac{92800}{89425}\cos 15^o 59'4''$$

$$\cos(x - \varphi) = 1,03774 \cos 15^o 59'4''$$

$$x - \varphi = \pm 3^o 57' + k.360^o$$

$$x_1 = \varphi + 3^o 57' + k.360^o = 19^o 56'4'' + k.360^o$$

$$x_2 = \varphi - 3^o 57' + k.360^o = 12^o 2'4'' + k.360^o.$$

6. Hitunglah x dari persamaan $5 \sin x + \operatorname{tg} x = 1$.

Penyelesaian:.........

$$5 \sin x + \operatorname{tg} x = 1$$

$$5 \sin x + \frac{\sin x}{\cos x} = 1$$

$$5 \sin x \cos x + \sin x = \cos x$$

$$5 \sin x \cos x + \sin x - \cos x = 0 ....(*)$$

Umpama $\sin x - \cos x = y$

$$\sin^2 x - 2 \sin x \cos x + \cos^2 x = y^2$$

$$1 - 2 \sin x \cos x = y^2$$

$$2 \sin x \cos x = 1 - y^2$$

$$\sin x \cos x = \frac{1 - y^2}{2}$$

persamaan (*) menjadi:

$$\frac{5(1 - y)^2}{2} + y = 0$$

$$5 - 5y^2 + 2y = 0$$

$$5y^2 - 2y - 5 = 0$$

$$y_{1.2} = \frac{2 \pm \sqrt{4 + 100}}{10}$$

$$= \frac{2 \pm \sqrt{104}}{10}$$

$$= \frac{2 \pm 10,19804}{10}$$

$y_1 = 1,21980$

$y_2 = -0,81980$

$(\sin 2x)_1 = 1 - y^2 = -048791$

$(\sin 2x)_2 = 1 - y^2 = 0,32793.$

dengan mempergunakan daftar logaritma,

untuk $\sin 2x = -0,48791$: $(2x)_1 = -150^\circ 47' 48'' + k.360^\circ$ kw II

$x_1 = -75^\circ 23' 54'' + k.180^\circ$

$(2x)_2 = 389^\circ 12' 12'' + k\ 360^\circ$ kw IV

$x_2 = 194^\circ 36' 6'' + k.180^\circ$

untuk $\sin 2x = 0,32793$: $(2x)_3 = 19^\circ 8' 36'' + k.360^\circ$ kw I

$x_3 = 9^\circ 34' 18'' + k.180^\circ$

$(2x)_4 = 170^\circ 25' 42'' + k.360^\circ$?

$x_4 = 85^\circ 12' 51'' + k.180^\circ$

7. Htunglah x dari persamaan $2 \sin 2x + 6 \cos^2 x = 5$.

Penyelesaian:

$2 \sin 2x + 6 \cos^2 x = 5$

$2 \sin 2x + 6 \cdot \frac{1 + \cos 2x}{2} = 5$

$2 \sin 2x + 3 + 3 \cos 2x = 5$

$3 \cos 2x + 2 \sin 2x = 2$

$3(\cos 2x + \frac{2}{3} \sin 2x) = 2$

$3(\cos 2x + \mathrm{tg}\,\varphi \sin 2x) = 2$ $\quad \begin{cases} \mathrm{tg}\,\varphi = \frac{2}{3} \\ \varphi = 33^\circ 41' 24''. \end{cases}$

$3(\cos 2x + \frac{\sin \varphi}{\cos \varphi} \sin 2x) = 2$

$3 \cos(2x - \varphi) = 2 \cos \varphi$

$\cos(2x - \varphi) = \frac{2 \cos \varphi}{3}$

$\cos(2x - \varphi) = \frac{2 \cdot 0,83205}{3} = 0,5547$

$2x - \varphi = 56^\circ 18'35" + k.360^\circ$

$2x_1 = 89^\circ 59'59" + k.360^\circ$

$x_1 = 45^\circ + k.180^\circ$

$2x - \varphi = 303^\circ 41'25" + k.360^\circ$

$2x_2 = 336^\circ 82'49" + k.360^\circ$

$x_2 = 168^\circ 41'25" + k.180^\circ$

8. Tentukan batas-batas p agar $p \sin x - (p - 1)\cos x = \sqrt{5}$ mempunyai penyelesaian.

Jawab:

Syarat: $p \sin x - (p - 1)\cos x = \sqrt{5}$, mempunyai penyelesaian:

$c^2 \leq a^2 + b^2.$

$(\sqrt{5})^2 < p^2 + (p - 1)^2$

$5 < p^2 + p^2 - 2p + 1$

$2p^2 - 2p - 4 \geq 0$

$p^2 - p - 2 \geq 0$

$(p + 1)(p - 2) \geq 0$

Jadi $p < -1$ dan $p > 2$

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<------|        |------->
  + +  |  - -   |  + +
------------------------
      -1        2
```

9. Tentukan batas-batas p agar $\sin(x - 30^\circ)\cos x = p \sin^2 x$ dapat diselesaikan.

Jawab:

$\sin(x - 30^\circ)\cos x = p \sin^2 x$

$2 \sin(x - 30^\circ)\cos x = 2p \sin^2 x$

$\sin(2x - 30^\circ) - \sin 30^\circ = p(1 - \cos 2x)$

$\sin 2x \cos 30^\circ - \cos 2x \sin 30^\circ - \sin 30^\circ = p(1 - \cos 2x)$

$\frac{1}{2}\sqrt{3} \sin 2x - \frac{1}{2}\cos 2x - \frac{1}{2} + p \cos 2x = p$

$\frac{1}{2}\sqrt{3} \sin 2x + (p - \frac{1}{2})\cos 2x = p + \frac{1}{2}$

$\sqrt{3} \sin 2x + (2p - 1)\cos 2x = 2p + 1$

$(2p + 1)^2 \leq (\sqrt{3})^2 + (2p - 1)^2$

$4p^2 + 4p + 1 \leq 3 + 4p^2 - 4p + 1$

$8p \leq 3 \longrightarrow p < \frac{3}{8}.$

10. Tentukan himpunan penyelesaian:

$$3 \sin 2x + 4 \sin^2 x = 5 \operatorname{tg} x$$

untuk interval $0^\circ \leq x \leq 360^\circ$.

Penyelesaian:

$3 \sin 2x + 4 \sin^2 x = 5 \operatorname{tg} x$

$6 \sin x \cos x + 4 \sin^2 x = \dfrac{5 \sin x}{\cos x}$

$\sin x(6 \cos x + 4 \sin x - \dfrac{5}{\cos x}) = 0$

$\sin x = 0 \longrightarrow x_1 = 0^\circ + k.360^\circ$

$6 \cos x + 4 \sin x - \dfrac{5}{\cos x} = 0$

$6 \cos^2 x + 4 \sin x \cos x - 5 = 0$

$3(1 + \cos 2x) + 2 \sin 2x = 5$

$3 \cos 2x + 2 \sin 2x = 2$

$3 \cos 2x = 2 - 2 \sin 2x$

$(\cos^2 x - \sin^2 x) = 2(1 - \sin 2x)$

$3(\cos^2 x - \sin^2 x) = 2(\cos x - \sin x)^2$

$3(\cos x - \sin x)(\cos x + \sin x) = 2(\cos x - \sin x)^2$

$\longrightarrow \cos x - \sin x = 0$

$\operatorname{tg} x = 1$

$x = 45^\circ + k.180^\circ$

$3(\cos x + \sin x) = 2(\cos x - \sin x)$

$3 \cos x + 3 \sin x = 2 \cos x - 2 \sin x$

$5 \sin x = -\cos x$

$5 \operatorname{tg} x = -1$

$\operatorname{tg} x = -0,2$

$\operatorname{tg}(-x) = 0,2 \longrightarrow -x = 11^\circ 19' + k.180^\circ$

$x = 168^\circ 41' + k.180^\circ$

$\therefore$ Himpunan penyelesaian:

$\{0^\circ, 45^\circ, 168^\circ 41', 180^\circ, 225^\circ, 248^\circ 41', 360^\circ\}$.

<u>Soal-Soal</u>:

Hitunglah x dari persamaan di bawah ini:

1. $3 \operatorname{tg}^2 x + 4 \sin^2 x = 2$ dalam $0^\circ \leq x \leq 360^\circ$.
2. $3 + \cos 4x = 0 \cos 2x$.

3. $\sin x \sin(x - 36^\circ) = 1.1 \cos^2 x$
4. $\sin^2 x + 4 \sin x \cos x - 7 \cos^2 x$.
5. $5 \sin x + \mathrm{tg}\, x = 1$.
6. $\cos 3x - 2 \cos^2 x = 0$.
7. $\mathrm{tg}\, x + \mathrm{tg}(x - 45^\circ) = 2$.
8. Tentukanlah batas-batas p agar
   $4 \sin^3 x - p \sin x \cos x = 1 - 2 \cos^2 x$.
9. Hitunglah p agar persamaan $\sin(x - 30^\circ)\cos x = p \sin^2 x$ dapat diselesaikan.
10. Tentukanlah batas-batas p agar
    $p \cos^2 x + (p - 1)\sin 2x = 2 \cos x$.
11. $3 \cos x + 7 \sin x = 6$.
12. $\cos 4x + (\propto_3)\sin 4x = 2$.
13. $10001 \cos x + 30145 \sin x = 9999$.
14. $\cos x \cos 19^\circ 47' 58'' + \sin x\, \mathrm{tg}\, 19^\circ 47' 58'' = 1$.
15. $2 \sin 2x + 6 \cos^2 x = 5$.
16. $\sin x \sin(x - 36^\circ) = 1.1 \cos^2 x$.
17. $7\, \mathrm{tg}\, x - 6\, \mathrm{cotg}\, x = 1{,}45$.
18. $2 \sin^2 x + 5 \sin x \cos x + 10 \cos^2 x = 7{,}92$.
19. $(1 - \sqrt{3})\sin x + (1 + \sqrt{3})\cos x = \sec x$.
20. $\mathrm{cotg}\, x - \mathrm{tg}\, x = \sin x + \cos x$.
21. $\sin x + 5 \sin x \cos x + \cos x = \pm 1$.
22. $2 \cos 3x + 2 \cos 2x = 1$.
23. $2 \cos 4x + 2\sin 4x + 2 \sin 2x + \mathrm{tg}\, x \sin^2 2x = 2 \cos^2 x$.
24. $\sin x + \mathrm{cotg}\, x + \mathrm{cosec}\, x = 0$.
25. $\cos x + \cos 3x + \sin 5x + \sin 3x = \cos(45^\circ - 4x)$.

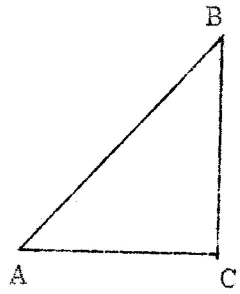
SEGITIGA

Uraian

Menurut bentuknya segitiga ada 3 macam:

1. Segitiga siku-siku.
2. Segitiga lancip : - segitiga sama kaki
   - segitiga sama sisi.
3. Segitiga tumpul.

Segitiga siku-siku ABC.



$\angle C = \gamma = 90^\circ$

$\alpha + \beta + \gamma = 180^\circ$

$\alpha + \beta = 90^\circ$

$a = c \sin\alpha$

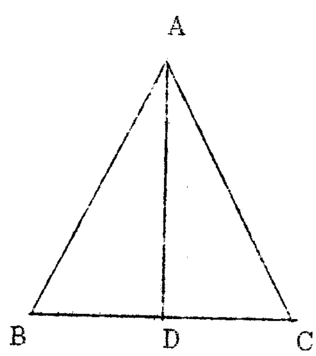
$b = c \sin\beta$

Gambar 1. Segitiga siku-siku ABC.

$$\left.\begin{aligned} a &= c \cos\beta \\ b &= c \cos\alpha \\ a &= b \operatorname{cotg}\alpha \\ b &= a \operatorname{tg}\beta \\ a &= b \operatorname{cotg}\beta \\ b &= a \operatorname{cotg}\alpha \end{aligned}\right\}$$ (PWijdenes, 1953. hal.166)

$a^2 + b^2 = c^2$ (dalil Pythagoras)

Segitiga sama kaki ABC.



$\alpha + \beta + \gamma = 180^\circ$

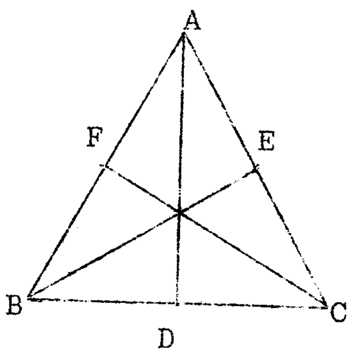
$\beta = \gamma$

$b = c$

AD merupakan garis tinggi, garis bagi dan garis berat dari titik sudut A.

Gambar 2. Segitiga sama kaki.

Segitiga sama sisi ABC.



$\alpha + \beta + \gamma = 180^\circ$

$\alpha = \beta = \gamma = 60^\circ$

AD = BE = CE

merupakan garis tinggi, garis berat dan garis bagi dari titik-titik sudut A, B dan C.

Gambar 3. Segitiga sama sisi.

<u>Contoh-contoh</u>:

1. Dari $\triangle$ ABC siku-siku ($\gamma = 90^\circ$), diketahui c = 13,57 dan $\beta = 37^\circ 5'20''$. Hitunglah unsur-unsur yang lain.

   Jawab:

   $\triangle$ ABC siku-siku ($\gamma = 90^\circ$)

   $\beta = 37^\circ 5'20''$

   $\alpha = 90^\circ - 37^\circ 5'20''$

   $= 52^\circ 54'40''$.

   $a = c \sin \alpha$

   $a = 13{,}57 \sin 52^\circ 54'40$

   $= 13{,}57$.

2. Dari $\triangle$ ABC, diketahui $\text{tg}\,\beta + \text{tg}\,\gamma = \sec\beta \sec\gamma$.

   Selidikilah bentuk $\triangle$ ABC itu.

   Jawab:

   $\text{tg}\,\beta + \text{tg}\,\gamma = \sec\beta \sec\gamma$.

   $$\frac{\sin\beta}{\cos\beta} + \frac{\sin\gamma}{\cos\gamma} = \frac{1}{\cos\beta} \cdot \frac{1}{\cos\gamma}$$

   $$\frac{\sin\beta \cos\gamma + \cos\beta \sin\gamma}{\cos\beta \cos\gamma} = \frac{1}{\cos\beta \cos\gamma}$$

   $\sin(\beta + \gamma) \cos\beta \cos\gamma = \cos\beta \cos\gamma$

   $\sin(\beta + \gamma) = 1$

$(\beta + \alpha) = 90^\circ$

$\gamma = 90^\circ$

$\therefore$ $\triangle$ ABC siku pada $\angle C$ ($\gamma = 90^\circ$).

3. Dari $\triangle$ ABC, diketahui $\sin\alpha = \text{cotg}\,\alpha\,(1 + \cos\alpha)$.
Selidikilah bentuk $\triangle$ ABC itu.
Jawab:

$\sin\alpha = \text{cotg}\,\gamma\,(1 + \cos\alpha)$

$\sin\alpha = \dfrac{\cos\gamma}{\sin\gamma}(1 + \cos\alpha)$

$\sin\alpha = \dfrac{\cos\gamma + \cos\gamma\cos\alpha}{\sin\gamma}$

$\cos\gamma + \cos\gamma\cos\alpha = \sin\gamma\sin\alpha$

$\cos\gamma + \cos\alpha\cos\gamma - \sin\alpha\sin\gamma = 0$

$\cos\gamma + \cos(\alpha + \gamma) = 0$

$-\cos\beta = -\cos\gamma$ atau

$\cos\beta = \cos\gamma$

$\beta = \gamma$

$\therefore$ Bentuk $\triangle$ ABC adalah sama kaki.

__Soal-Soal__:

1. Dari $\triangle$ ABC($\gamma = 90^\circ$), diketahui:
   a). c = 18,73 dan $\alpha = 10^\circ 13' 12''$, hitunglah a dan b.
   b). a = 31,4 dan c = 20,45, hitunglah α.
   c). a = 25,37 dan $\alpha = 15^\circ 35' 20''$, hitunglah b dan c.

2. Buktikanlah dalam $\triangle$ ABC ($\gamma = 90^\circ$) berlaku:
   a). $\sin \frac{1}{2}\alpha = \sqrt{\dfrac{c - b}{2c}}$
   b). $\text{tg}\,\frac{1}{2}\alpha = \sqrt{\dfrac{c - b}{c + b}}$
   c). Hitunglah α, jika b = 51,36 dan c = 51,6.

3. Selidikilah bentuk $\triangle$ ABC, jika diketahui:

a). $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$.

b). $\operatorname{tg}\beta + \operatorname{tg}\gamma = \sec\beta \sec\gamma$.

c). $\sin\alpha - \cos\alpha = \cos\beta - \sin\beta$.

d). $\operatorname{cotg} \frac{1}{2}\alpha + \operatorname{cotg} \frac{1}{2}\beta - \operatorname{cotg} \frac{1}{2}\alpha \cdot \operatorname{cotg} \frac{1}{2}\beta = -1$.

4. Buktikanlah dalam $\triangle$ ABC ($\gamma = 90^o$) berlaku:

a). $\dfrac{a + b}{\cos \frac{1}{2} (\alpha - \beta)} = c\sqrt{2}$.

b). $\operatorname{tg} \frac{1}{2} (\alpha - \beta) = \dfrac{a - b}{a + b}$

5. Selidikilah bentuk $\triangle$ ABC, jika diketahui:

a). $\sin\alpha = \operatorname{cotg}\gamma (1 + \cos\alpha)$.

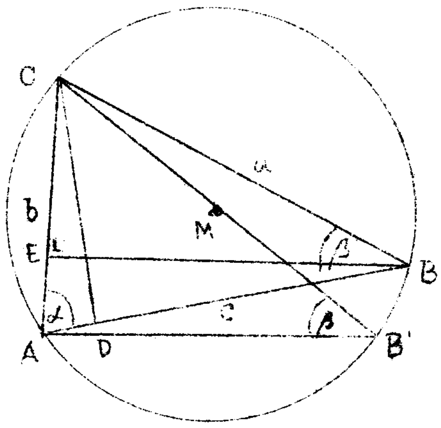
b). $\sin^2\gamma = \sin 2\alpha \sin 2\beta$.

c). $\cos\alpha \cos\beta = \sin^2 \frac{1}{2}\gamma$.

d). $\operatorname{tg} \frac{1}{2}\alpha = \dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha}$

e). $\cos 2(\beta - \gamma) - \cos 2\alpha = \cos 2 (\alpha - \gamma) - \cos 2\beta$.

DALIL SINUS DAN COSINUS PADA SEGITIGA



Gambar 4. Segitiga ABC dan lingkaran luarnya.

Perhatikan $\triangle$ ABC

CD = garis tinggi dari titik C.

BE = garis tinggi dari titik B.

$\triangle ACD \longrightarrow CD = b \sin\alpha$

$\triangle BCD \longrightarrow CD = a \sin\beta$

$b \sin\alpha = a \sin\beta$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \;\ldots\ldots\ldots\ldots\ldots(*)$$

$\triangle ABE \longrightarrow BE = c \sin\alpha$

$\triangle BCE \longrightarrow BE = a \sin\gamma$

$c \sin\alpha = a \sin\gamma$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \;\ldots\ldots\ldots\ldots\ldots(**)$$

Dari (*) dan (**) didapat:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \ldots\ldots\ldots\ldots(***)$$

$\angle CBA = \angle CB'A$

M = pusat ʘ luar $\triangle$ ABC.

Pada $\triangle$ BB'A $\longrightarrow \sin\beta = \frac{b}{2R}$

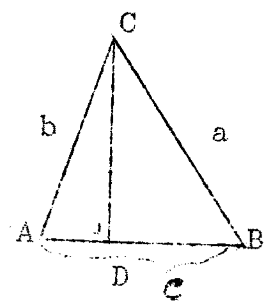
$$\frac{b}{\sin\beta} = 2R \ldots\ldots\ldots\ldots\ldots(****)$$

Dari (***) dan (****) didapat:

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R}$$ dalil sinus.

atau

$$\boxed{\begin{aligned} a &= 2R \sin\alpha \\ b &= 2R \sin\beta \\ c &= 2R \sin\gamma \end{aligned}}$$



Gambar 5. Segitiga ABC dan garis tinggi CD.

Perhatikan gambar 5.

$AD = b \cos \alpha$

$DB = c - b \cos \alpha$

$CD = b \sin \alpha$

Perhatikan $\triangle$ BCD

$BC^2 = CD^2 + BD^2$

$a^2 = (b \sin\alpha)^2 + (c - b \cos\alpha)^2$

$a^2 = b^2 \sin^2\alpha + c^2 - 2bc \cos \alpha + b^2 \cos^2\alpha$

$a^2 = b^2(\sin^2\alpha + \cos^2\alpha) + c^2 - 2bc \cos\alpha$

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos\alpha}$$ dalil cosinus.

Dengan cara yang sama didapat:

$$\boxed{\begin{aligned} b^2 &= c^2 + a^2 - 2ac \cos\beta \\ c^2 &= a^2 + b^2 - 2ab \cos\gamma \end{aligned}}$$

<u>Dalil-dalil yang dapat diturunkan dari dalil sinus.</u>

1. Dalil tangens dari Napier.

$$\frac{a+b}{a-b} = \frac{2R \sin\alpha + 2R \sin\beta}{2R \sin\alpha - 2R \sin\beta}$$

$$= \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$$

$$= \frac{\cancel{2} \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}}{\cancel{2} \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}}$$

$$= \operatorname{tg} \frac{\alpha+\beta}{2} \operatorname{cotg} \frac{\alpha-\beta}{2}$$

$$\boxed{\frac{a+b}{a-b} = \frac{\operatorname{tg} \frac{1}{2}(\alpha+\beta)}{\operatorname{tg} \frac{1}{2}(\alpha-\beta)}}$$

2. $$\frac{a+b}{c} = \frac{2R \sin\alpha + 2R \sin\beta}{2R \sin\gamma}$$

$$= \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}}{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}}$$

$$= \frac{\cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)}$$

$$= \frac{\sin \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(\alpha+\beta)}$$

$$\boxed{\frac{a+b}{c} = \frac{\cos \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}\gamma}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

$$\frac{a-b}{c} = \frac{2R \sin\alpha - 2R \sin\beta}{2R \sin\gamma}$$

$$= \frac{\sin\alpha - \sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2 \cancel{\cos \frac{\alpha+\beta}{2}} \sin \frac{\alpha-\beta}{2}}{2 \sin \frac{\alpha+\beta}{2} \cancel{\cos \frac{\alpha+\beta}{2}}}$$

$$\boxed{\frac{a-b}{c} = \frac{\sin \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}\gamma}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(**)$$

(*) dan (**) menurut Wijdenes (1953, hal.179) dinamakan dalil De Lambre.

Dalil-dalil yang diturunkan dari dalil cosinus

Dalil cosinus:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$2bc \cos \alpha = b^2 + c^2 - a^2$$

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$1 + \cos \alpha = 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$2 \cos^2 \tfrac{1}{2}\alpha = \frac{(b + c)^2 - a^2}{2bc}$$

$$2 \cos^2 \tfrac{1}{2}\alpha = \frac{(b + c + a)(b + c - a}{2bc}$$

$$= \frac{2.s.2(s - a)}{2bc}$$

$$\cos^2 \tfrac{1}{2}\alpha = \frac{s(s - a)}{bc}$$

$$\boxed{\cos^2 \tfrac{1}{2}\alpha = \sqrt{\frac{s(s - a)}{bc}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Dengan cara yang sama didapat:

$$\boxed{\begin{aligned} \cos^2 \tfrac{1}{2}\beta &= \sqrt{\frac{s(s - b)}{ac}} \\ \cos^2 \tfrac{1}{2}\gamma &= \sqrt{\frac{s(s - c)}{ab}} \end{aligned}}$$

Dari dalil cosinus di atas didapat pula:

$$1 - \cos \alpha = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$1 - \cos \alpha = 1 - \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$2\sin^2 \tfrac{1}{2}\alpha = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc}$$

$$2\sin^2 \tfrac{1}{2}\alpha = \frac{2(s-c)\ 2(s-b)}{2bc}$$

$$\sin^2 \tfrac{1}{2}\alpha = \frac{(s-b)(s-c)}{bc}$$

$$\boxed{\sin^2 \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}}} \quad \ldots\ldots\ldots\ldots\ldots (**)$$

Dengan cara yang sama didapat:

$$\sin^2 \tfrac{1}{2}\beta = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$\sin^2 \tfrac{1}{2}\gamma = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\operatorname{tg}^2 \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\operatorname{tg}^2 \tfrac{1}{2}\beta = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \quad \ldots\ldots\ldots\ldots\ldots (***)$$

$$\operatorname{tg}^2 \tfrac{1}{2}\gamma = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Menurut **Alders**, (1953, hal.87); (*), (**) dan (***) dinamakan dalil **Gauss**.

Contoh-Contoh:

1. Buktikanlah dalam segitiga berlaku:

$$\operatorname{tg} \tfrac{1}{2} (\alpha + 2\beta) = \frac{c + b}{c - b} \operatorname{tg} \tfrac{1}{2}\alpha$$

Bukti :

$$\operatorname{tg} \tfrac{1}{2} (\alpha + 2\beta) = \frac{c + b}{c - b} \operatorname{tg} \tfrac{1}{2}\alpha$$

$$= \frac{2R \sin \gamma + 2R \sin \beta}{2R \sin \gamma - 2R \sin \beta} \cdot \operatorname{tg} \tfrac{1}{2}\alpha$$

$$= \frac{\sin(\alpha + \beta) + \sin \beta}{\sin(\alpha + \beta) + \sin \beta} \cdot \operatorname{tg} \tfrac{1}{2}\alpha$$

$$= \frac{2 \sin \frac{\alpha + 2\beta}{2} \cancel{\cos \tfrac{1}{2}\alpha}}{2 \cos \frac{\alpha + 2\beta}{2} \cancel{\sin \tfrac{1}{2}\alpha}} \cdot \frac{\cancel{\sin \tfrac{1}{2}\alpha}}{\cancel{\cos \tfrac{1}{2}\alpha}}$$

$$= \frac{\sin \tfrac{1}{2} (\alpha + 2\beta)}{\cos \tfrac{1}{2} (\alpha + 2\beta)}$$

$\operatorname{tg} \tfrac{1}{2} (\alpha + 2\beta) = \operatorname{tg} \tfrac{1}{2} (\alpha + 2\beta)$ ——> terbukti.

2. Buktikanlah:

$$\frac{s - a}{b} = \frac{\cos \tfrac{1}{2}\alpha}{\cos \tfrac{1}{2}\beta} \sin \tfrac{1}{2}\gamma \qquad (s = \tfrac{1}{2} \text{ keliling } \triangle \text{ ABC})$$

$$\frac{b + c - a}{2b} = \frac{2R \sin \beta + 2R \sin \gamma - 2R \sin \alpha}{2 \cdot 2R \sin \beta}$$

$$= \frac{\sin \beta + \sin \gamma - \sin \alpha}{2 \sin \beta}$$

$$= \frac{2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} - \sin (\beta + \gamma)}{2 \sin \beta}$$

$$= \frac{2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} - 2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta + \gamma}{2}}{2 \sin \beta}$$

$$= \frac{\cancel{2} \sin \frac{\beta + \gamma}{2} \left[\cos \frac{\beta - \gamma}{2} - \cos \frac{\beta + \gamma}{2} \right]}{\cancel{2} \sin \beta}$$

$$= \frac{\cos\frac{1}{2}\alpha \cdot \cancel{2\sin\frac{1}{2}\beta}\sin\frac{1}{2}\gamma}{\cancel{2\sin\frac{1}{2}\beta}\cos\frac{1}{2}\beta}$$

$$= \frac{\cos\frac{1}{2}\alpha\sin\frac{1}{2}\gamma}{\cos\frac{1}{2}\beta}$$

$$= \frac{\cos\frac{1}{2}\alpha}{\cos\frac{1}{2}\beta}\sin\frac{1}{2}\gamma = \frac{\cos\frac{1}{2}\alpha}{\cos\frac{1}{2}\beta}\sin\frac{1}{2}\gamma$$

terbukti.

3. Buktikanlah dalam $\triangle$ ABC berlaku $\sum\frac{b-c}{a}\cos^2\frac{1}{2}\alpha = 0$.

Bukti:

$$\sum\frac{b-c}{a}\cos^2\frac{1}{2}\alpha = 0$$

$$\sum\frac{\cancel{2R}\sin\beta - \cancel{2R}\sin\gamma}{\cancel{2R}\sin\alpha}\cos^2\frac{1}{2}\alpha =$$

$$\sum\frac{\sin\beta - \sin\gamma}{\sin\alpha}\cos^2\frac{1}{2}\alpha =$$

$$\sum\frac{2\cos\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}}{\sin(\beta+\gamma)}\cos^2\frac{1}{2}\alpha =$$

$$\sum\frac{\cancel{2}\cancel{\cos\frac{\beta+\gamma}{2}}\sin\frac{\beta-\gamma}{2}}{\cancel{2}\sin\frac{\beta+\gamma}{2}\cancel{\cos\frac{\beta+\gamma}{2}}}\cos^2\frac{1}{2}\alpha =$$

$$\sum\frac{\sin\frac{\beta-\gamma}{2}}{\cos\frac{1}{2}\alpha}\cos^2\frac{1}{2}\alpha =$$

$$\sum\sin\left(\frac{\beta-\gamma}{2}\right)\cos\frac{1}{2}\alpha =$$

$$\sum\sin\left(\frac{1}{2}\beta - \frac{1}{2}\gamma\right)\cos\frac{1}{2}\alpha =$$

$\sin(\frac{1}{2}\beta - \frac{1}{2}\gamma)\cos\frac{1}{2}\alpha + \sin(\frac{1}{2}\gamma - \frac{1}{2}\alpha)\cos\frac{1}{2}\beta + \sin(\frac{1}{2}\alpha - \frac{1}{2}\beta)\cos\frac{1}{2}\gamma =$
$(\sin\frac{1}{2}\beta\cos\frac{1}{2}\gamma - \cos\frac{1}{2}\beta\sin\frac{1}{2}\gamma)\cos\frac{1}{2}\alpha + (\sin\frac{1}{2}\gamma\cos\frac{1}{2}\alpha - \cos\frac{1}{2}\gamma$
$\sin\frac{1}{2}\alpha)\cos\frac{1}{2}\beta + (\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta - \cos\frac{1}{2}\alpha\sin\frac{1}{2}\beta)\cos\frac{1}{2}\gamma = 0 = 0$

4. Dari $\triangle$ ABC, diketahui:

$$\frac{\cos^3 \frac{1}{2}\alpha \cos \frac{1}{2}(\beta - \gamma)}{\sin \beta} = \frac{b + c}{4b}$$

Selidikilah bentuk $\triangle$ ABC itu.

Penyelesaian:

$$\frac{\cos^3 \frac{1}{2}\alpha \cos \frac{1}{2}(\beta - \gamma)}{\sin\beta} = \frac{2R \sin\beta + \cancel{2R} \sin\gamma}{4 . \cancel{2R} \sin\beta}$$

$$\frac{\cos^3 \frac{1}{2}\alpha \cancel{\cos \frac{1}{2}(\beta - \gamma)}}{\cancel{\sin\beta}} = \frac{\cancel{2.}\sin \frac{1}{2}(\beta + \gamma) \cancel{\cos \frac{1}{2}(\beta - \gamma)}}{2.4\cancel{.\sin\beta}}$$

$$\cos^3 \frac{1}{2}\alpha = \frac{\sin \frac{1}{2}(\beta + \gamma)}{2}$$

$2 \cos^3 \frac{1}{2}\alpha = \cos \frac{1}{2}\alpha$

$2 \cos^3 \frac{1}{2}\alpha - \cos \frac{1}{2}\alpha = 0$

$\cos \frac{1}{2}\alpha(2\cos^2 \frac{1}{2}\alpha - 1) = 0$

$\cos \frac{1}{2}\alpha = 0 \longrightarrow \frac{1}{2}\alpha = 90^\circ$

$\alpha = 180^\circ$ (tidak mungkin).

$2 \cos^2 \frac{1}{2}\alpha - 1 = 0$

$2 \cos^2 \frac{1}{2} = 1$

$\cos^2 \frac{1}{2}\alpha = \frac{1}{2}$

$\cos^2 \frac{1}{2}\alpha = \sqrt{\frac{1}{2}}$

$\cos^2 \frac{1}{2}\alpha = \frac{1}{2}\sqrt{2}$

$\frac{1}{2}\alpha = 45^\circ \longrightarrow \alpha = 90^\circ$

$\therefore$ $\triangle$ ABC siku di A.

Soal-Soal:

Dalam $\triangle$ ABC, R adalah jari-jari lingkaran luar $\triangle$ ABC, $s = \frac{1}{2}$ kelilingnya.

Buktikanlah:

1. $\operatorname{tg} \gamma = \dfrac{c \sin \alpha}{b - c \cos \alpha}$

2. $\dfrac{\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta}{\sin \frac{1}{2}\gamma} = \dfrac{s}{c}$

3. $\operatorname{tg} \frac{1}{2}\alpha \operatorname{tg} \frac{1}{2}\beta = \dfrac{s - c}{s}$

4. $\dfrac{s - a}{s - b} = \dfrac{\operatorname{cotg} \frac{1}{2}\alpha}{\operatorname{cotg} \frac{1}{2}\beta}$

5. $\sum (b + c) \cos \alpha = 2s$

6. $R = \dfrac{s}{\sin \alpha + \sin \beta + \sin \gamma} = \dfrac{s}{4 \cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma}$

7. $\dfrac{a^2 - b^2}{c^2} = \dfrac{\sin(\alpha - \beta)}{\sin \gamma}$

9. $\sum \dfrac{b \cos \gamma + c \cos \beta}{\sin \alpha} = 6R$

10. $a \cos \alpha + b \cos \beta = c \cos (\alpha - \beta)$

11. $\dfrac{a \cos \alpha - b \cos \beta}{b \cos \alpha - a \cos \beta} = \cos \gamma$

12. $\sum \dfrac{b - c}{a} \cos \frac{1}{2}\alpha = 0$

13. $\sum \dfrac{a - 2c \cos \beta}{c \sin \beta} = 0$

14. Dalam $\triangle$ ABC, sisi $a + c = 2b$, buktikanlah:

a). $\cos \frac{1}{2} (\alpha - \gamma) = 2 \sin \frac{1}{2} \beta$

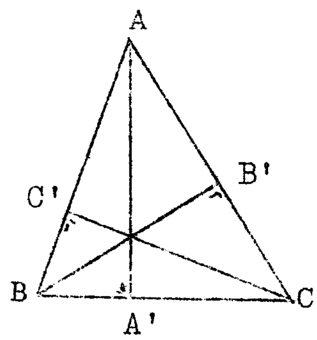
b). $\text{tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\gamma = \frac{1}{3}$.

15. Dalam $\triangle$ ABC, sudut $\gamma = 2\alpha$, buktikanlah:

a). $c^2 = a(a + b)$

b). $\cos \alpha = \dfrac{a + b}{c}$.

GARIS TINGGI PADA SEGITIGA



adalah garis tinggi pada $\triangle$ ABC.
H dinamakan titik tinggi $\triangle$ ABC.

Gambar 6 segitiga ABC ketiga garis tingginya.

$$
\begin{aligned}
ha &= \sin \gamma \\
&= 2b \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma \\
&= 2b \sqrt{\frac{(s-a)(s-b)}{ab}} \cdot \sqrt{\frac{s(s-c)}{ab}} \\
&= \frac{2b}{ab} \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}
$$

$$\boxed{ha = \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}}$$

Dengan cara yang sama didapat:

$$\boxed{\begin{aligned} hb &= \frac{2}{b} \sqrt{s(s-a)(s-b)(s-c)} \\ hc &= \frac{2}{c} \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}}$$

Perhatikan $\triangle$ BA'H:

$$
\begin{aligned}
\text{tg } \angle B_1 &= \frac{HA'}{BA'} \\
HA' &= BA' \text{tg } \angle B_1 \\
&= c \cos \beta \text{ tg}(90^\circ - \gamma) \\
&= c \cos \beta \text{ cotg} \\
&= 2R \sin \gamma \cos \beta \frac{\cos \gamma}{\sin \gamma}
\end{aligned}
$$

$$\boxed{HA' = 2R \cos \beta \cos \gamma}$$

Dengan cara yang sama didapat:

$$\boxed{\begin{aligned} HB' &= 2R\cos\alpha\cos\gamma \\ HC' &= 2R\cos\alpha\cos\beta \end{aligned}}$$

Perhatikan $\triangle AC'H$:

$$\cos\angle A_1 = \frac{AC'}{HA}$$

$$\begin{aligned} HA &= \frac{AC'}{\cos\angle A_1} \\ &= \frac{b\cos\alpha}{\cos(90^o-\beta)} \\ &= \frac{b\cos\alpha}{\sin\beta} \\ &= \frac{2R\cancel{\sin\beta}\cos\alpha}{\cancel{\sin\beta}} \end{aligned}$$

$$\boxed{HA = 2R\cos\alpha}$$

Dengan cara yang sama didapat:

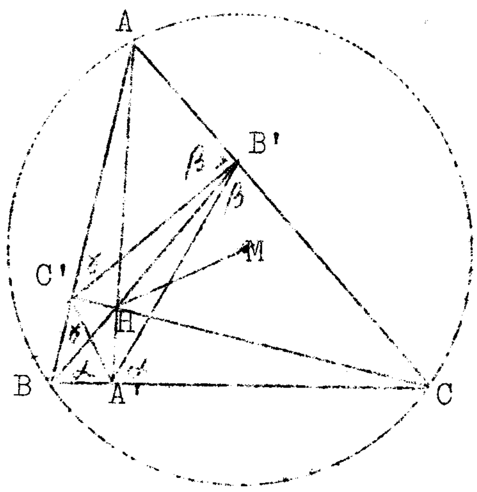
$$\boxed{\begin{aligned} HB &= 2R\cos\beta \\ HC &= 2R\cos\gamma \end{aligned}}$$

$$\begin{aligned} HA'.\ HA &= 4R^2\cos\alpha\cos\beta\cos\gamma \\ HB'.\ HB &= 4R^2\cos\alpha\cos\beta\cos\gamma \\ HC'.\ HC &= 4R^2\cos\alpha\cos\beta\cos\gamma \end{aligned}$$

$\therefore$ $HA'.HA = HB'.HB = HC'.HC = 4R^2\cos\alpha\cos\beta\cos\gamma$

M = pusat lingkaran luar $\triangle$ ABC.

$$\begin{aligned} U(H) &= -8R^2\cos\alpha\cos\beta\cos\gamma \\ U(H) &= HM^2 - R^2 \\ HM^2 &= U(H) + R^2 \\ HM^2 &= -8R^2\cos\alpha\cos\beta\cos\gamma + R^2 \\ HM^2 &= R^2(1 - 8\cos\alpha\cos\beta\cos\gamma) \end{aligned}$$



Segitiga ABC, segitiga titik-kaki dan lingkaran luarnya.

Gambar 7.

Perhatikan Gambar 7:

$\triangle A'B'C'$ dinamakan segitiga titik kaki.

$$B'C' = a_v$$
$$A'C' = b_v$$
$$A'B' = c_v$$

Perhatikan $\triangle$ B'C'C:

$$B'C' : \sin(90^\circ - \alpha) = CB' : \sin(90^\circ - \gamma)$$
$$B'C' : \cos\alpha = a\cos\gamma : \cos\gamma$$
$$B'C' : \cos\alpha = a : 1.$$

$$\boxed{B'C' = a\cos\alpha}$$ atau $$\boxed{a_v = a\cos\alpha}$$

Dengan cara yang sama didapat pula:

$$\boxed{\begin{array}{l} b_v = b\cos\beta \\ c_v = c\cos\gamma \end{array}}$$

Perhatikan $\triangle$ A'B'C':

$$\angle A' = \angle B'A'C' = 180^\circ - 2\alpha$$
$$\angle B' = \angle A'B'C' = 180^\circ - 2\beta$$
$$\angle C' = \angle B'C'A' = 180^\circ - 2\gamma$$

R_v = jari-jari lingkaran luar $\triangle$ A'B'C'

$$a_v = 2R_v \sin A'$$

$$a \cos\alpha = 2R_v \sin(180^o - 2\alpha)$$
$$a \cos\alpha = 2R_v \sin 2\alpha$$
$$2R \sin\alpha \cos\alpha = 2R_v \sin 2\alpha$$
$$R \sin 2\alpha = 2R_v \sin 2\alpha$$
$$R = 2R_v$$

$$\boxed{R_v = \frac{1}{2} R}$$

Contoh:

1. Buktikanlah dalam segitiga ABC berlaku: $HA = a \operatorname{cotg}\alpha$.

   Bukti:

$$HA = a \operatorname{cotg}\alpha$$
$$= 2R \sin\alpha \frac{\cos\alpha}{\sin\alpha}$$
$$= 2R \cos\alpha$$
$$HA = HA \longrightarrow \text{terbukti.}$$

2. Buktikanlah:

$$a^2 + HA^2 = b^2 + HB^2 = c^2 + HC^2 = 4R^2$$

   Bukti:

$$a^2 + (2R \cos\alpha)^2 = (2R \sin\alpha)^2 + (2R \cos\alpha)^2$$
$$= 4R^2\sin^2\alpha + 4R^2\cos^2\alpha = 4R^2$$
$$b^2 + HB^2 = (2R \sin\beta)^2 + (2R \cos\beta)^2$$
$$4R^2\sin^2\beta + 4R^2\cos^2\beta = 4R^2$$
$$c^2 + HC^2 = (2R \sin\gamma)^2 + (2R \cos\gamma)^2$$
$$4R^2\sin^2\gamma + 4R^2\cos^2\gamma = 4R^2$$
$$\therefore a^2 + HA^2 = b^2 + HB^2 = c^2 + HC^2 = 4R^2 \longrightarrow \text{terbukti.}$$

<u>Soal-Soal</u>:

Buktikanlah dalam $\triangle$ ABC berlaku:

1. $A_v = R \sin 2\alpha = HA \sin \alpha$.

2. $AB'.BC'.CA' = AC'.BA'.CB' = a_v\ b_v\ c_v$.
   Buktikanlah $U(H) = -4Rr_v$.

3. r_v adalah jari-jari lingkaran dalam $\triangle A'B'C'$, buktikanlah:
$$r_v = 2R \cos\alpha \cos\beta \cos\gamma .$$

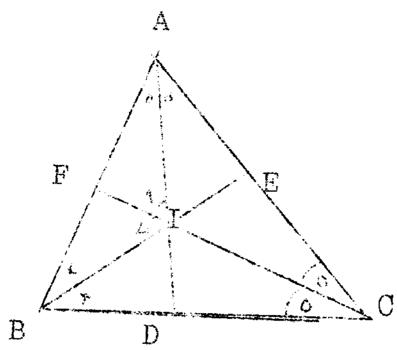
4. Buktikanlah: $HM^2 = 9R^2 - a^2 - b^2 - c^2$.

5. Jika H adalah titik tinggi, O(M.R) adalah lingkaran luar $\triangle$ ABC, buktikanlah: $HA^2 + HB^2 + HC^2 - HM^2 = 3R^2$.

6. Jika $HM = \frac{1}{2} a$, $\alpha = 90^o$.
   Buktikanlah $tg\beta \cdot tg\gamma = 9$.

GARIS BAGI PADA SEGITIGA

Uraian:



Menurut Wijdenes (1953, hal. 210) garis bagi dalam segitiga ABC adalah : $AD = d_\alpha$

$BE = d_\beta$

$CF = d_\gamma$.

AD, BE dan CF berpotongan di I

I = pusat lingkaran dalam $\triangle$ ABC.

Gambar 8. Segitiga dan ketiga garis baginya

Perhatikan $\triangle$ ABD : (*Gambar 8*).

$AD : \sin\beta = AB : \sin \angle D_1$ (dalil sinus)

$d_\alpha : \sin\beta = c : \sin(\frac{1}{2}\alpha + \gamma)$

$d_\alpha \cdot \sin(\frac{1}{2}\alpha + \gamma) = c \sin\beta$.

$$\boxed{d = \frac{c \sin\beta}{\sin(\frac{1}{2}\alpha + \gamma)}}$$

dengan cara yang sama didapat:

$$\boxed{\begin{array}{l} d = \dfrac{a \sin\gamma}{\sin(\frac{1}{2}\beta + \alpha)} \\[2em] d = \dfrac{b \sin\alpha}{\sin(\frac{1}{2}\gamma + \beta)} \end{array}}$$

Selanjutnya menurut Alders, d_α , d_β dan d_γ juga dapat dicari dengan pertolongan rumus luas segitiga, yaitu:

Luas $\triangle$ ABC = luas $\triangle$ ABD + luas $\triangle$ ACD

Luas $\triangle$ ABC = $\frac{1}{2}$ AB.AD $\sin \frac{1}{2}\alpha$ + $\frac{1}{2}$ AD.AC $\sin \frac{1}{2}\alpha$

$\frac{1}{2}$ AB.AC$\sin\alpha = \frac{1}{2}$ c.$d_\alpha \sin \frac{1}{2}\alpha + \frac{1}{2} d_\alpha$ .b $\sin \frac{1}{2}\alpha$

$\frac{1}{2}$ b.c $\sin\alpha = \frac{1}{2} d_\alpha \sin \frac{1}{2}\alpha (b + c)$

$bc \sin\alpha = d \quad \sin \frac{1}{2}\alpha (b + c)$

$2bc \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha = d_\alpha \sin \frac{1}{2}\alpha (b + c)$

$$\boxed{d_\alpha = \frac{2bc \cos \frac{1}{2}\alpha}{b + c}}$$

dengan cara yang sama didapat:

$$\boxed{\begin{aligned} d_\beta &= \frac{2ac \cos \frac{1}{2}\beta}{a + c} \\ d_\gamma &= \frac{2ab \cos \frac{1}{2}\gamma}{a + b} \end{aligned}}$$

Perhatika $\triangle$ AIB (*Gambar 8*)

$AI : \sin \frac{1}{2}\beta = AB : \sin \angle I_{1.2}$

$AI : \sin \frac{1}{2}\beta = c : \sin\{180^\circ - (\frac{1}{2}\alpha + \frac{1}{2}\beta)\}$

$AI : \sin \frac{1}{2}\beta = c : \sin(\frac{1}{2}\alpha + \frac{1}{2}\beta)$

$AI : \sin \frac{1}{2}\beta = c : \sin(90^\circ - \frac{1}{2}\gamma)$

$AI : \sin \frac{1}{2}\beta = c : \cos \frac{1}{2}\gamma$

$AI : \cos \frac{1}{2}\gamma = c \sin \frac{1}{2}\beta$

$AI : \cos \frac{1}{2}\gamma = 2R \sin\gamma \sin \frac{1}{2}\beta$

$= 4R \sin \frac{1}{2}\gamma \cos \frac{1}{2}\gamma \sin \frac{1}{2}\beta$

$$AI = \frac{4R \sin \frac{1}{2}\gamma \cos \frac{1}{2}\gamma \sin \frac{1}{2}\beta}{\cos \frac{1}{2}\gamma}$$

$$\boxed{AI = 4R \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma}$$

Dengan cara yang sama didapat:

$$\boxed{\begin{aligned} BI &= 4R \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\gamma \\ CI &= 4R \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \end{aligned}}$$

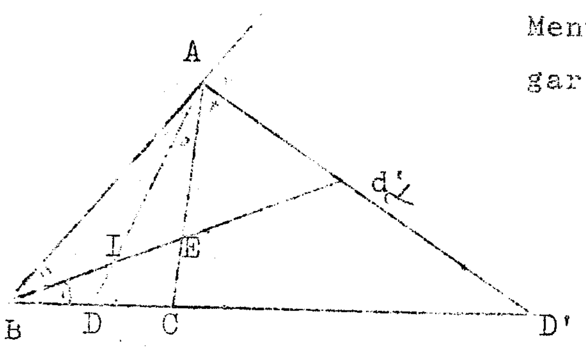
Perhatikan $\triangle$ BID (*Gambar 8*)

$ID : \sin \frac{1}{2}\beta = BI : \sin \angle D_1$

$ID : \sin \frac{1}{2}\beta = 4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\gamma : \sin(\frac{1}{2}\alpha + \gamma)$

$ID . \sin(\frac{1}{2}\alpha + \gamma) = 4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma$

$$\boxed{\begin{aligned} ID &= \frac{4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma}{\sin(\frac{1}{2}\alpha + \gamma)} \\ IE &= \frac{4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma}{\sin(\frac{1}{2}\beta + \alpha)} \\ IF &= \frac{4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma}{\sin(\frac{1}{2}\gamma + \beta)} \end{aligned}}$$



Menurut Wijdenes,(1953, hal. garis bagi luar $\triangle$ ABC adalah.

$AD' = d'_\alpha$

$BE' = d'_\beta$

$CF' = d'_\gamma$

Gambar 9. Segitiga dan garis bagi $\angle A$.

Menurut Alders, (1969, hal. 48) dengan mempergunakan rumus luas segitiga dapat ditentukan d' .

Perhatikan gambar 9:

Luas $\triangle$ ABC = Luas $\triangle$ ABC + Luas $\triangle$ ACD'

$\frac{1}{2}$ AB.AD' $\sin(90^\circ + \frac{1}{2}\alpha) = \frac{1}{2}$ AB.AC $\sin\alpha + \frac{1}{2}$ AC.AD' $\sin(90^\circ - \frac{1}{2}\alpha)$

$$\tfrac{1}{2}c\,d'_\alpha \cos \tfrac{1}{2}\alpha = \tfrac{1}{2}\,bc\,\sin\alpha + \tfrac{1}{2}\,b.d'_\alpha \cos \tfrac{1}{2}\alpha$$

$$d'_\alpha \cos \tfrac{1}{2}\alpha(c - b) = 2bc \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha$$

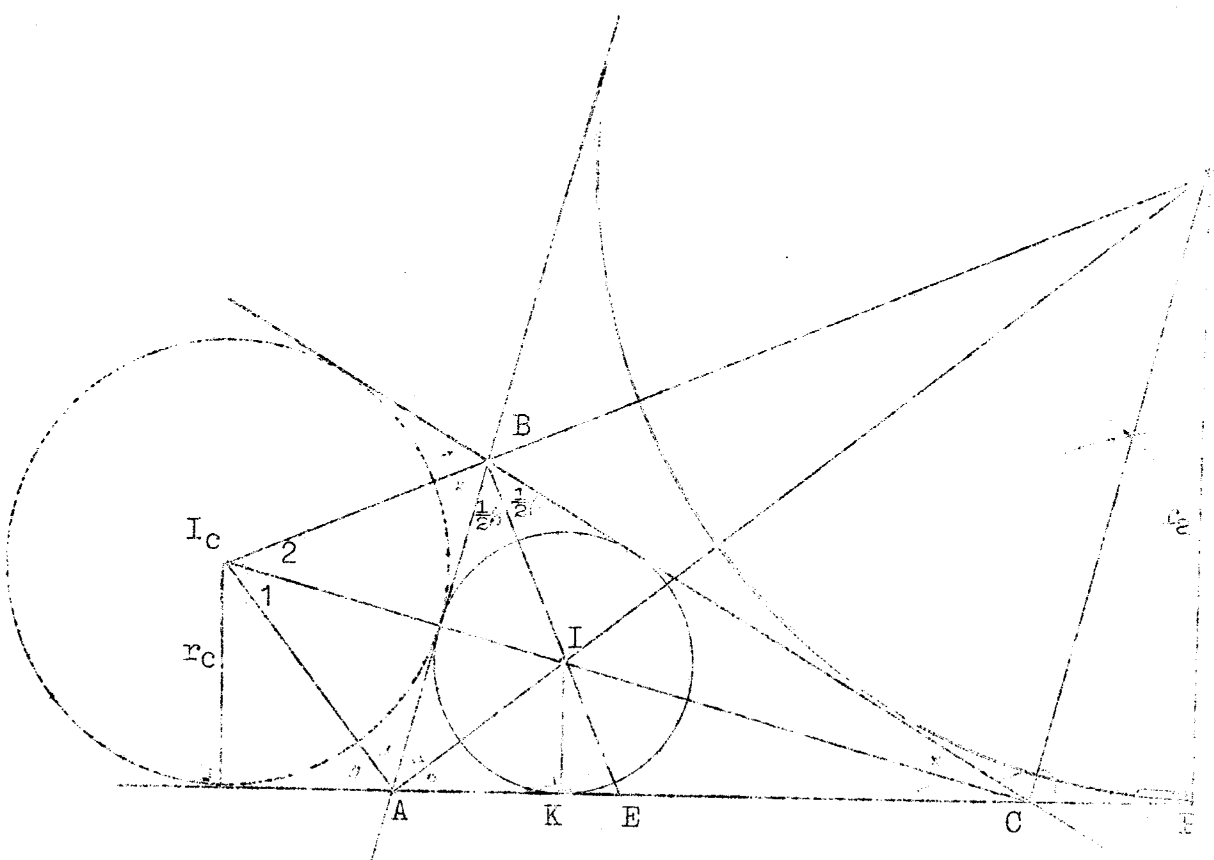
$$d'_\alpha (c - b) = 2bc \sin \tfrac{1}{2}\alpha$$

$$d'_\alpha = \frac{2bc \sin \frac{1}{2}\alpha}{c - b} \qquad (c > b)$$

$$d'_\alpha = \frac{2bc \sin \frac{1}{2}\alpha}{|\,b - c\,|}$$

$$d'_\beta = \frac{2bc \sin \frac{1}{2}\beta}{|\,a - c\,|}$$

$$d'_\gamma = \frac{2bc \sin \frac{1}{2}\gamma}{|\,a - b\,|}$$



Gambar 10. Segitiga ABC beserta lingkaran-lingkaran yang menyinggungdi luar sisi a dan c.

Perhatikan gambar 10:

I = titik pusat lingkaran dalam $\triangle$ ABC

IK = r = jari-jari lingkaran dalam $\triangle$ ABC

I_a = pusat lingkaran yang menyinggung sisi a di luar

I_b = pusat lingkaran yang menyinggung sisi b di luar

I_c = pusat lingkaran yang menyinggung sisi c di luar.

Perhatikan $\triangle$ BI_cA (gambar 10).

$BI_c = \sin(90^o - \frac{1}{2}\alpha) = AB : \sin \angle Ic_{1-r}$

$BI_c : \cos \frac{1}{2}\alpha = c : \sin\{180^o - (90^o - \frac{1}{2}\alpha) - (90^o - \frac{1}{2}\beta)\}$

$BI_c : \cos \frac{1}{2}\alpha = c : \sin(\frac{1}{2}\alpha + \frac{1}{2}\beta)$

$BI_c : \cos \frac{1}{2}\alpha = c : \sin(90^o - \frac{1}{2}\gamma)$

$BI_c . \cos \frac{1}{2}\gamma = c \cos \frac{1}{2}\alpha$

$BI_c \cos \frac{1}{2}\gamma = 2R \sin\gamma \cos \frac{1}{2}\alpha$

$$BI_c = \frac{4R \sin \frac{1}{2}\gamma \cancel{\cos \frac{1}{2}\gamma} \cos \frac{1}{2}\alpha}{\cancel{\cos \frac{1}{2}\gamma}}$$

$$\boxed{BI_c = 4R \sin \frac{1}{2}\gamma \cos \frac{1}{2}\alpha}$$

dengan cara yang sama didapat:

$$\boxed{\begin{array}{l} CI_a = 4R \sin \frac{1}{2}\alpha \cos \frac{1}{2}\beta \\ AI_b = 4R \sin \frac{1}{2}\beta \cos \frac{1}{2}\gamma \end{array}}$$

Perhatikan $\triangle$ AI_aC (Gambar 10).

$AI_a : \sin(90^o + \frac{1}{2}\gamma) = AC : \sin \angle I_{a1}$

$AI_a : \cos \frac{1}{2}\gamma = b : \sin\{180^o - (90^o + \frac{1}{2}\gamma) - \frac{1}{2}\alpha\}$

$AI_a : \cos \frac{1}{2}\gamma = b : \sin\{90^o + (\frac{1}{2}\alpha + \frac{1}{2}\gamma)\}$

$AI_a : \cos \frac{1}{2}\gamma = b : \sin \frac{1}{2}\beta$

$AI_a : \sin \frac{1}{2}\beta = 2R \sin\beta \cos \frac{1}{2}\gamma$

$AI_a : \sin \frac{1}{2}\beta = 4R \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma$

$$\boxed{AI_a = 4R \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma}$$

dengan cara yang sama didapat:

$$BI_b = 4R \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\gamma$$

$$CI_c = 4R \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta$$

Perhatikan $\triangle\ II_aC$ (Gambar 10).

$II_a : \sin 90^\circ = CI : \sin \angle I_{a1}$

$II_a : 1 = 4R \sin \tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta : \sin \tfrac{1}{2}\beta$

$II_a.\sin \tfrac{1}{2}\beta = 4R \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta$

$$II_a = 4R \sin \tfrac{1}{2}\alpha$$

$$II_b = 4R \sin \tfrac{1}{2}\beta$$

$$II_c = 4R \sin \tfrac{1}{2}\gamma$$

Perhatikan $\triangle$ AIK (Gambar 10).

$\sin \tfrac{1}{2}\alpha = \frac{IK}{AI}$

$\sin \tfrac{1}{2}\alpha = \frac{r}{AI}$

$r = AI.\sin \tfrac{1}{2}\alpha$

$$r = 4R \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma$$

Perhatikan $\triangle\ AI_aP$ (Gambar 10).

$\sin \tfrac{1}{2}\alpha = \frac{I_aP}{AI_a}$

$\sin \tfrac{1}{2}\alpha = \frac{r_a}{4R \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma}$

$$r_a = 4R \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma$$

$$r_b = 4R \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\gamma$$

$$r_c = 4R \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta$$

Dengan memperhatikan gambar 10; I_bI_c , I_cI_a dan I_aI_b juga dapat ditentukan.

Contoh:

1. Pada $\triangle$ ABC, I = pusat lingkaran dalam segitiga, $s = \frac{1}{2}$ kelilingnya, buktikanlah bahwa:

$$IA = \frac{s - a}{\cos \frac{1}{2}\alpha} = \frac{bc}{s} \cos \frac{1}{2}\alpha$$

Bukti:

a). $$IA = \frac{s - a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c - a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{b + c - a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{2R \sin\beta + 2R \sin\gamma - 2R \sin\alpha}{2 \cos \frac{1}{2}\alpha}$$

$$= \frac{R[\sin\beta + \sin\gamma - \sin(\beta + \gamma)]}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R[2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} - 2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta + \gamma}{2}]}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R[2 \sin \frac{\beta + \gamma}{2} (\cos \frac{\beta - \gamma}{2} - \cos \frac{\beta + \gamma}{2})]}{\cos \frac{1}{2}\alpha}$$

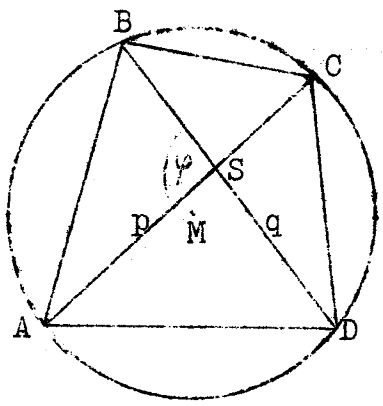
$$= \frac{R[2 \cancel{\cos \frac{1}{2}\alpha}\ 2 \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma]}{\cancel{\cos \frac{1}{2}\alpha}}$$

$IA = 4R \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma$

IA = IA. terbukti.

b). $$IA = \frac{bc}{s} \cos \frac{1}{2}\alpha$$

SEGI EMPAT TALI BUSUR



Gambar 16.

Segi-4 ABCD adalah segi-4 tali-busur.

$AB = a$

$BC = b$

$CD = c$

$DA = d$

diagonal $AC = p$ dan diagonal $BD = q$, kedua diagonal berpotongan di S.

Sudut antara p dan q adalah φ .

Perhatikan $\triangle ABD$:

$$BD^2 = AB^2 + AD^2 - 2AB.AD\cos A$$
$$q^2 = a^2 + d^2 - 2ad\cos A \ldots\ldots\ldots\ldots\ldots(*)$$

Perhatikan $\triangle BCD$:

$$BD^2 = BC^2 + CD^2 - 2BC.CD\cos C$$
$$q^2 = b^2 + c^2 - 2bc\cos C$$
$$q^2 = b^2 + c^2 + 2bc\cos A \ldots\ldots\ldots\ldots\ldots(**)$$

Dari (*) dan (**) didapat:

$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 + 2bc\cos A$$
$$2bc\cos A + 2ad\cos A = a^2 + d^2 - b^2 - c^2$$
$$\cos A(2bc + 2ad) = a^2 + d^2 - b^2 - c^2$$
$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2bc + 2ad}$$

$$1 + \cos A = 1 + \frac{a^2 - b^2 - c^2 + d^2}{2bc + 2ad}$$
$$= \frac{2bc + 2ad + a^2 - b^2 - c^2 + d^2}{2(bc + ad)}$$
$$= \frac{(a + d)^2 - (b - c)^2}{2(bc + ad)}$$
$$2\cos^2 \tfrac{1}{2} A = \frac{(a + d + b - c)(a + d - b + c)}{2(bc + ad)}$$
$$= \frac{2(s - c).\cancel{2}(s - b)}{\cancel{2}(bc + ad)}$$

dengan menggunakan dalil sinus kembali didapat pula:

$$\frac{a}{p} \cdot \frac{p}{c} \cdot \frac{c}{q} \cdot \frac{q}{a} = 1$$

$$\frac{\sin C_1}{\sin B} \cdot \frac{\sin D}{\sin A_1} \cdot \frac{\sin B_2}{\sin C} \cdot \frac{\sin A}{\sin D_2} = 1$$

atau

$$\boxed{\sin A \sin B_2 \sin C_1 \sin D = \sin A_1 \sin B \sin C \sin D_2.}$$

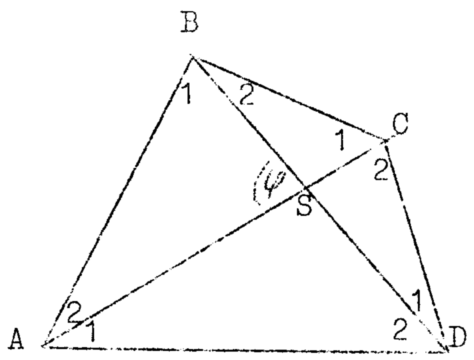
dan

$$\frac{c}{d} \cdot \frac{d}{a} \cdot \frac{a}{b} \cdot \frac{b}{c} = 1$$

$$\boxed{\sin A_1 \sin B_1 \sin C_1 \sin D_1 = \sin A_2 \sin B_2 \sin C_2 \sin D_2.}$$

SEGI EMPAT

Uraian:



Segi-4 ABCD adalah segi-4 sembarangan.

AB = a

BC = b

CD = c

DA = d

diagonal AC = p

diagonal BD = q

Gambar 15.

Diagonal AC dan BD berpotongan di S. $\angle ASB = \varphi$.

Perhatikan $\triangle ACD$:

Menurut dalil sinus:

$$\frac{p}{\sin D_{1.2}} = \frac{d}{\sin C_2}$$

$$\frac{d}{p} = \frac{\sin c_2}{\sin D_{1.2}}$$

Perhatikan $\triangle ABC$:

$$\frac{p}{\sin B_{1.2}} = \frac{a}{\sin C_1}$$

$$\frac{p}{a} = \frac{\sin B_{1.2}}{\sin C_1}$$

Pada $\triangle ABD$ didapat pula:

$$\frac{d}{\sin B_1} = \frac{a}{\sin D_2}$$

$$\frac{a}{d} = \frac{\sin D_2}{\sin B_1}$$

$\frac{d}{p} \times \frac{p}{a} \times \frac{a}{d} = 1$

$\frac{\sin C_2}{\sin D_{1.2}} \times \frac{\sin {}_{1.2}}{\sin C_1} \times \frac{\sin D_2}{\sin B_1}$

$\frac{\sin C_2 . \sin B . \sin D_2}{\sin D . \sin C_1 . \sin B_1} = 1$

atau

$$\boxed{\sin B . \sin C_2 . \sin D_2 = \sin B_1 . \sin C_1 \sin D.}$$

$$\frac{\sin(\beta - w)}{\sin w} = \frac{\sin^2\beta}{\sin\alpha \sin\gamma}$$

$$\frac{\sin\beta \sin w - \cos\beta \sin w}{\sin w} = \frac{\sin^2\beta}{\sin\alpha \sin\gamma}$$

kemudian ruas kiri dan kanan sama-sama diagi dengan sin se-hingga didapat:

$$\frac{\cos w - \cotg\beta \sin w}{\sin w} = \frac{\sin\beta}{\sin\alpha \sin\gamma}$$

$$\frac{\cos w - \cotg\beta \sin w}{\sin w} = \frac{\sin(\alpha + \gamma)}{\sin\alpha \sin\gamma}$$

$$\cotg w - \cotg\beta = \frac{\sin\alpha \cos\gamma + \cos\alpha \sin\gamma}{\sin\alpha \sin\gamma}$$

$$\cotg w - \cotg\beta = \cotg\gamma + \cotg\alpha$$

$$\cotg w = \cotg\beta + \cotg\gamma + \cotg\alpha$$

atau

$$\boxed{\cotg w = \cotg\alpha + \cotg\beta + \cotg\gamma\ .}$$

Soal-Soal:

Dalam $\triangle ABC$, M', I' dan H' adalah proyeksi dari M, I dan H. (Gambar 13).

Buktikanlah:

1. $NN' = \frac{1}{2}R\cos(\beta - \gamma)$.
2. $H'M' = R\sin(\beta - \gamma)$.
3. $H'I' = 4R\sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma \sin\frac{1}{2}(\beta - \gamma)$.
4. $I'M' = 2R\sin\frac{1}{2}\alpha \sin\frac{1}{2}(\beta - \gamma)$.
5. $I'N' = R\left|\sin\beta - \sin\gamma - \frac{1}{2}\sin(\beta - \gamma)\right|$
6. $AN^2 = \frac{1}{4}R^2(1 + 8\cos\alpha \sin\beta \sin\gamma)$.
7. $MI_a^2 = R^2 + 2Rr_a$.
8. $HI_a^2 = 2r_a^2 - 4R^2\cos\alpha\cos\beta\cos\gamma$.

Pada $\triangle HIM$, IN adalah garis berat, maka:

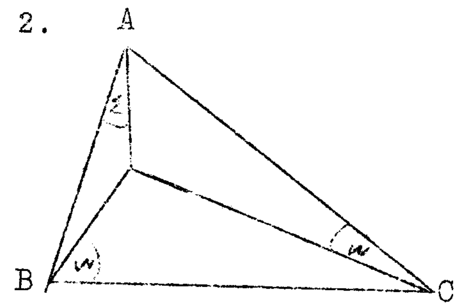
$$IN^2 = \tfrac{1}{2} HI^2 + \tfrac{1}{2} MI^2 - \tfrac{1}{4} HM^2$$

$$IN^2 = r^2 - 2R^2 \cos\alpha \cos\beta \cos\gamma + \tfrac{1}{2}R - Rr - \tfrac{1}{4}R^2 + 2R^2 \cos\alpha \cos\beta \cos\gamma$$

$$= \tfrac{1}{4}R^2 - Rr + r^2$$

$$= (\tfrac{1}{2}R - r)^2$$

$$IN = \tfrac{1}{2}R - r.$$

2. 

Perhatikan $\triangle ABC$ (gambar 14).
P didalam $\triangle ABC$, sehingga:
$\angle PAB = \angle PBC = \angle PCA = w$.
Menurut ***Wijdeness*** (1963,hal.222), titik P dinamakan titik **Brocard**.

Gambar 14: Segitiga dan titik Brocard.

Buktikanlah: $\text{cotg } w = \text{cotg}\,\alpha + \text{cotg}\,\beta + \text{cotg}\,\gamma$.

Bukti:

$\angle PAC = \alpha - w$.

$\angle APC = \beta + \gamma = 180^\circ - \alpha$.

Jadi $\angle BPA = 180^\circ - \beta$ dan

$\angle CPB = 180^\circ - \gamma$.

Menurut dalil sinus:

$$\frac{AP}{\sin w} = \frac{b}{\sin\alpha} \quad \text{dan} \quad \frac{AP}{\sin(\beta - w)} = \frac{c}{\sin\beta}$$

$$\left.\begin{aligned} AP &= \frac{b \sin w}{\sin\alpha} \\ AP &= \frac{c \sin(\beta - w)}{\sin\beta} \end{aligned}\right\}$$

$$\frac{c \sin(\beta - w)}{\sin\beta} = \frac{b \sin w}{\sin\alpha}$$

$$c \sin(\beta - w)\sin\alpha = b \sin w . \sin\beta .$$

$$\frac{\sin(\beta - w)}{\sin w} = \frac{b \sin\beta}{c \sin\alpha}$$

$$\frac{\sin(\beta - w)}{\sin w} = \frac{2R \sin\beta . \sin\beta}{2R \sin\gamma \sin\alpha}$$

<u>Penyelesaian</u>:

$HM^2 = R^2(1 - 8\cos\alpha\cos\beta\cos\gamma) \longrightarrow$ dalil.

Perhatikan $\triangle MBI$:

$$BI - 4R\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\gamma\,.$$

$BM = R$

$\angle MBI = \frac{1}{2}\,|\alpha - \gamma\,|.$

Menurut dalil cosinus:

$MI^2 = BM^2 + BI^2 - 2\,BM.Bi\cos\angle MBI$

$MI^2 = R^2 + 16R^2\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\gamma - 8R^2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\gamma$
$\quad\cos\tfrac{1}{2}(\alpha - \gamma).$

$\quad = R^2 - 8R^2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha\,\{\cos\tfrac{1}{2}(\alpha - \gamma) - 2\sin\tfrac{1}{2}\alpha$
$\quad\sin\tfrac{1}{2}\gamma\,\}.$

$\quad = R^2 - 8R^2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}(\alpha - \gamma)$

$\quad = R^2 - 8R^2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma$

$\quad = R^2 - 2R.4\,R\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma$

$$\boxed{MI^2 = R^2 - 2\,Rr.}$$

Perhatikan $\triangle HBI$:

HI dapat dihitung dengan menggunakan dalil cosinus:

$HB = 2R\cos\beta\,.$

$BI = 4R\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\gamma\,.$

$\angle HBI = \frac{1}{2}(\alpha - \gamma)$, maka:

$HI^2 = HB^2 + BI^2 - 2HB.BI\cos\angle HBI$

$HI^2 = 4R^2\cos^2\gamma + 16R^2\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\gamma - 16R^2\sin\tfrac{1}{2}\alpha\cos\beta$
$\quad\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}(\alpha - \gamma).$

$\quad = 4R^2(\cos^2\beta + 4\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\gamma - 4\sin\tfrac{1}{2}\alpha\cos\beta\sin\tfrac{1}{2}\gamma \times$
$\quad\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\gamma - 4\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\gamma\cos\beta\,.$

$\quad = 4R^2\{-\cos\beta\cos(\alpha + \gamma) - \cos\beta\sin\alpha\sin\gamma + 4\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\gamma$
$\quad(1 - \cos\beta).$

$\quad = 4R^2(-\cos\alpha\cos\beta\cos\gamma + 8\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma)$

$\quad = 32R^2\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma - 4R^2\cos\alpha\cos\beta\cos\gamma\,.$

$$\boxed{HI^2 = 2r^2 - 4R^2\cos\alpha\cos\beta\cos\gamma\,.}$$

Perhatikan $\triangle ACG$:

Menurut dalil cosinus: $AG^2 = AC^2 + CG^2 - 2\,AC.CG \cos \angle ACG$

$$(2\,Z_2)^2 = b^2 + c^2 - 2\,bc \cos(180^o - \alpha)$$

$$4\,Z_a^2 = b^2 + c^2 + 2\,bc \cos\alpha\,.$$

$$\boxed{Z_a^2 = \tfrac{1}{4}(b^2 + c^2 + 2\,bc \cos\alpha).}$$

Dengan mempergunakan dalil cosinus kembali, maka:

$2\,bc \cos \quad = b^2 + c^2 - a^2$, sehingga didapat:

$Z_a^2 = \frac{1}{4}(b^2 + c^2 + b^2 + c^2 - a^2)$, atau

$$\boxed{Z_a^2 = \tfrac{1}{2}(b^2 + c^2) - \tfrac{1}{4}a^2.}$$

Perhatikan kembali gambar 1 dan 2:

$\angle ADC = \varphi_a$, $\angle BEA = \varphi_b$ dalam $\angle CFB = \varphi_c$

$AA' = h_a$

$A'C - BA' = ZA'D$ (lihat gambar 2).

$$\text{cotg}(180^o - \varphi_a) = \frac{A'D}{h_a} = \frac{1}{2}\left(\frac{A'C}{h_a} - \frac{BA'}{h_a}\right)$$

$$-\text{cotg}\,\varphi_a = \tfrac{1}{2}(\text{cotg}\,\gamma - \text{cotg}\,\beta)$$

$$-\text{cotg}\,\varphi_a = \tfrac{1}{2}(\text{cotg}\,\beta - \text{cotg}\,\gamma)$$

Dengan cara yang sama didapat:

$$\text{cotg}\,\varphi_b = \tfrac{1}{2}(\text{cotg}\,\gamma - \text{cotg}\,\alpha)$$

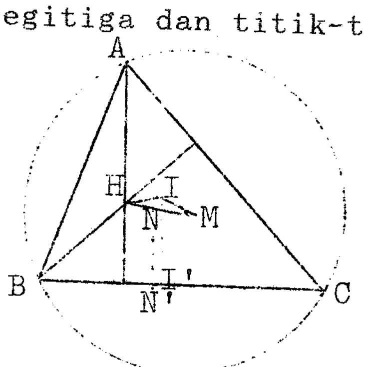
$$\text{cotg}\,\varphi_c = \tfrac{1}{2}(\text{cotg}\,\alpha - \text{cotg}\,\beta)$$

seterusnya didapat:

$$\boxed{\text{cotg}\,\varphi_a + \text{cotg}\,\varphi_b + \text{cotg}\,\varphi_c = 0.}$$

Contoh-contoh:

1. Segitiga dan titik-titik istimewa pada $\triangle ABC$ (gambar 13)



Gambar 13.

M = titik pusat lingkaran luar segitiga ABC.

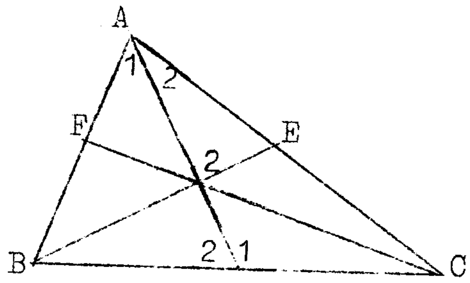
I = titik pusat lingkaran dalam segitiga ABC.

N = titik tengah HM.

Hitunglah: MI^2 , HI^2 dan IN.

GARIS BERAT PADA SEGITIGA

Uraian:



Pada segitiga ABC (gambar 11).

$AD = Z_a$

$BE = Z_b$

$CF = Z_c$

adalah garis berat $\triangle ABC$, ketiga garis berat ini berpotongan di Z.

Gambar 11: Segitiga dan garis beratnya.

Menurut Marcus , (1971 , hal.216): $AZ : ZD = 2 : 1$

$BZ : ZE = 2 : 1$

$CZ : ZF = 2 : 1$

$\angle ADC = \angle D_1$ dan $\angle ADB = \angle D_2$.

Pada $\triangle ABD \longrightarrow AB : \sin D_2 = BD : \sin \alpha_1$

$c : \sin D_2 = \frac{1}{2} a : \sin \alpha_1$

$$\sin D_2 = \frac{c \sin \alpha_1}{\frac{1}{2} a}$$

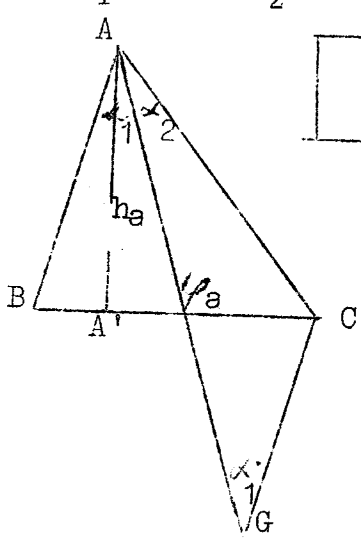
Pada $\triangle ABC \longrightarrow AC : \sin D_1 = CD : \sin \alpha_2$

$b : \sin D_2 = \frac{1}{2} a : \sin \alpha_2$

$$\sin D_1 = \frac{b \sin \alpha_2}{\frac{1}{2} a}$$

$\sin D_1 = \sin D_2 \longrightarrow c \sin \alpha_1 = b \sin \alpha_2$ atau

$$\boxed{\sin \alpha_1 : \sin \alpha_2 = b : c.}$$



Perhatikan gambar 12:

$AD = Z_a$ adalah garis berat dari titik sudut A.

Perpanjangan AD dengan DG, sehingga $AD = DG = Z_a$, maka $AG = 2\ Z_a$.

$CG = AB = C$

$\angle AGC = \alpha_1$ dan $\angle ACG = 180^o - \alpha$.

Gambar 12: Segitiga ABC dan perpanjangan garis berat AD = DG.

4. $\frac{1}{r} - \frac{1}{r_c} = \frac{1}{R \sin\alpha \sin\beta}$

5. $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

6. $tg^2 \frac{1}{2}\alpha = \frac{2r_a}{r_b\ r_c}$

7. $rr_a + r_b\ r_c = bc$.

8. $\sum r_b\ r_c = S^2$.

9. $AI_b .\ AI_c = bc$

10. $4R.II_c + II_b .\ II_a = I_bI_c .\ I_c.I_a$.

11. Buktikanlah:

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma} = \frac{1}{r}.$$

Bukti:

$r = S \text{ tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= \frac{a+b+c}{2} \text{ tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= \frac{2R \sin\alpha + 2R \sin\beta + 2R \sin\gamma}{2} \text{ tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= R(\sin\alpha + \sin\beta + \sin\gamma) \text{ tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= R[2 \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \sin(\alpha+\beta)] \text{tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= R[2 \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2 \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}] \text{tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= R[2 \sin\frac{\alpha+\beta}{2}(\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2})] \text{tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$

$= R[4 \cos\frac{1}{2}\gamma \cancel{\cos} \frac{1}{2}\alpha \cos\frac{1}{2}\beta] \frac{\sin\frac{1}{2}\alpha \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma}{\cos\frac{1}{2}\alpha \cos\frac{1}{2}\beta \cos\frac{1}{2}\gamma}$

$= 4R \sin\frac{1}{2}\alpha \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma$

$r = r$, terbukti.

6. Buktikanlah $r_a - r = 4R \sin^2 \frac{1}{2}\alpha$.

Bukti:

$r_a - r = 4R \sin^2 \frac{1}{2}\alpha$

$4R \sin\frac{1}{2}\alpha \cos\frac{1}{2}\beta \cos\frac{1}{2}\gamma - 4R \sin\frac{1}{2}\alpha \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma =$

$4R \sin\frac{1}{2}\alpha (\cos\frac{1}{2}\beta \cos\frac{1}{2}\gamma - \sin\frac{1}{2}\beta \sin\frac{1}{2}\gamma) =$

$4R \sin\frac{1}{2}\alpha \cos(\frac{1}{2}\beta + \frac{1}{2}\gamma) =$

$4R \sin\frac{1}{2}\alpha \sin\frac{1}{2}\alpha =$

$4R \sin^2\frac{1}{2}\alpha = 4R \sin^2\frac{1}{2}\alpha$, terbukti.

<u>Soal-Soal</u>:

Buktikanlah:

1. $\sum r_a = 3r + \sum a \text{ tg } \frac{1}{2}\alpha$.
2. $r = S \text{ tg } \frac{1}{2}\alpha \text{ tg } \frac{1}{2}\beta \text{ tg } \frac{1}{2}\gamma$.
3. $\text{cotg } \frac{1}{2}\alpha + \text{cotg } \frac{1}{2}\beta + \text{cotg } \frac{1}{2}\gamma = \frac{S^3}{r_a\, r_b\, r_c}$

$= 4R \sin \frac{1}{2}\alpha [\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma + \cos(\frac{1}{2}\beta + \frac{1}{2}\gamma)]$

$= 4R \sin \frac{1}{2}\alpha (\cancel{\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma} + \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma -$

$- \cancel{\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma})$

$r_a = 4R \sin \frac{1}{2}\alpha \cos \frac{1}{2}\beta \cos \frac{1}{2}\alpha$

$r_a = r_a$, terbukti.

3. Buktikanlah:

$$\frac{1}{r_a} + \frac{1}{r_b} = \frac{1}{R \sin\alpha \sin\beta}$$

Bukti:

$$\frac{1}{r_a} + \frac{1}{r_b} = \frac{1}{R \sin\alpha \sin\beta}$$

$$\frac{1}{4R \sin\alpha \cos \frac{1}{2}\beta \cos \frac{1}{2}\alpha} + \frac{1}{4R \sin \frac{1}{2}\beta \cos \frac{1}{2}\alpha \cos \frac{1}{2}\gamma} =$$

$$\frac{\sin \frac{1}{2}\beta \cos \frac{1}{2}\alpha + \cos \frac{1}{2}\beta \sin \frac{1}{2}\alpha}{4R \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma} =$$

$$\frac{\sin (\frac{1}{2}\alpha + \frac{1}{2}\beta)}{R \sin\alpha \sin\beta \cos \frac{1}{2}\gamma} =$$

$$\frac{\cancel{\cos \frac{1}{2}\gamma}}{R \sin\alpha \sin\beta \cancel{\cos \frac{1}{2}\gamma}} =$$

$$\frac{1}{R \sin\alpha \sin\beta} = \frac{1}{R \sin\alpha \sin\beta}$$, terbukti.

4. Bukikanlah $r_b + r_c = 4R \cos^2 \frac{1}{2}\alpha$.

Bukti:

$r_b + r_c = 4R \cos^2 \frac{1}{2}\alpha$

$4R \cos^2 \frac{1}{2}\beta \cos \frac{1}{2}\alpha \cos \frac{1}{2}\gamma + 4R \sin \frac{1}{2}\gamma \cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta =$

$4R \cos \frac{1}{2}\alpha (\sin \frac{1}{2}\beta \cos \frac{1}{2}\gamma + \cos \frac{1}{2}\beta \sin \frac{1}{2}\gamma) =$

$4R \cos \frac{1}{2}\alpha [\sin (\frac{1}{2}\beta + \frac{1}{2}\gamma)] =$

$4R \cos \frac{1}{2}\alpha . \cos \frac{1}{2}\alpha =$

$4R \cos^2 \frac{1}{2}\alpha = 4R \cos^2 \frac{1}{2}\alpha$, terbukti.

5. Buktikanlah $r = S \operatorname{tg} \frac{1}{2}\alpha \operatorname{tg} \frac{1}{2}\beta \operatorname{tg} \frac{1}{2}\gamma$.

$$= \frac{bc \cos \frac{1}{2}\alpha}{\frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} c}$$

$$= \frac{2bc \cos \frac{1}{2}\alpha}{a + b + c}$$

$$= \frac{2\ bc \cos \frac{1}{2}\alpha}{2R \sin\alpha + 2R \sin\beta + 2R \sin\gamma}$$

$$= \frac{bc \cos \frac{1}{2}\alpha}{R[\sin(\beta+\gamma) + \sin\beta + \sin\gamma]}$$

$$= \frac{2\ bc \cos \frac{1}{2}\alpha}{R[2 \sin \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2} + 2 \sin \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2}]}$$

$$= \frac{2\cancel{R} \sin\beta \cdot 2R \sin\gamma \cos \frac{1}{2}\alpha}{\cancel{R}[2 \sin \frac{\beta+\gamma}{2} (\cos \frac{\beta+\gamma}{2} + \cos \frac{\beta-\gamma}{2})]}$$

$$= \frac{\cancel{4}R \sin\beta \sin\gamma \cancel{\cos \frac{1}{2}\alpha}}{\cancel{2} \cancel{\cos \frac{1}{2}\alpha} \cdot \cancel{2} \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma}$$

$$= \frac{4R \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta \sin \frac{1}{2}\gamma \cos \frac{1}{2}\gamma}{\cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma}$$

$IA = 4R \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma$

$IA = IA$. terbukti.

Dari a) dan b), didapat:

$$IA = \frac{s - a}{\cos \frac{1}{2}\alpha} = \frac{bc}{s} \cos \frac{1}{2}\alpha$$

2. Buktikanlah $r_a = r + a \operatorname{tg} \frac{1}{2}\alpha$.

Bukti:

$r_a = r + a \operatorname{tg} \frac{1}{2}\alpha$

$= 4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma + 2R \sin\alpha \operatorname{tg} \frac{1}{2}\alpha$

$= 4R \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma + 4R \sin \frac{1}{2}\alpha \cancel{\cos \frac{1}{2}\alpha} \cdot \frac{\sin \frac{1}{2}\alpha}{\cancel{\cos \frac{1}{2}\alpha}}$

$= 4R \sin \frac{1}{2}\alpha [\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma + \sin \frac{1}{2}\alpha]$

$$\cos^2 \tfrac{1}{2} A = \frac{(s + b)(s + c)}{bc + ad}$$

$$\boxed{\cos^2 \tfrac{1}{2} A = \sqrt{\frac{(s + b)(s + c)}{bc + ad}}}$$

$$1 - \cos A = 1 - \frac{a^2 - b^2 - c^2 + d^2}{2bc + 2ad}$$

$$= \frac{2bc + 2ad - a^2 + b^2 + c^2 - d^2}{2bc + 2ad}$$

$$= \frac{(b + c + a - d)(b + c - a + d)}{2bc + 2ad}$$

$$= \frac{2.(s - d).\ \cancel{2}(s - a)}{\cancel{2}(bc + ad)}$$

$$2 \sin^2 \tfrac{1}{2} A = \frac{2(s - a)(s - d)}{bc + ad}$$

$$\sin^2 \tfrac{1}{2} A = \frac{(s - a)(s - d)}{bc + ad}$$

$$\boxed{\sin^2 \tfrac{1}{2} A = \sqrt{\frac{(s - a)(s - d)}{bc + ad}}}$$

$$\operatorname{tg} \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} A}{\cos \tfrac{1}{2} A}$$

$$\boxed{\operatorname{tg} \tfrac{1}{2} A = \sqrt{\frac{(s - a)(s - d)}{(s - b)(s - c)}}}$$

$$\begin{aligned}
\angle \text{segi-4 } ABCD &= \angle_{\triangle} ABD + \angle_{\triangle} BCD \\
&= \tfrac{1}{2}\, AB.AD \sin A + \tfrac{1}{2}\, BC.CD \sin C \\
&= \tfrac{1}{2}\, ad \sin A + \tfrac{1}{2}\, bc \sin C \\
&= \tfrac{1}{2}\, ad \sin A + \tfrac{1}{2}\, bc \sin A \\
&= \tfrac{1}{2} \sin A\ (ad + bc)
\end{aligned}$$

$$= \frac{1}{2}.2 \sin A \frac{1}{2} A \cos A\ (ad + bc)$$

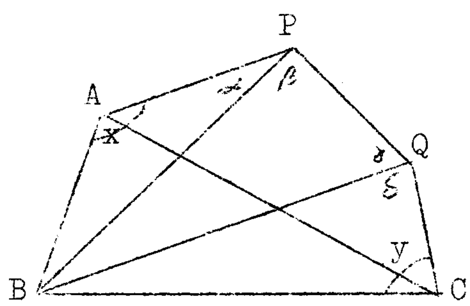
$$= \sin \frac{1}{2} A + \cos \frac{1}{2} A\ (ad + bc)$$

$$= \sqrt{\frac{(s-a)(s-d)}{bc + ad}} \sqrt{\frac{(s-b)(s-c)}{bc + ad}}\ (ad + bc)$$

$$= (ad + bc) \sqrt{\frac{(s-a)(s-b)(s-c)(s-d)}{(bc + ad)^2}}$$

$$\boxed{\angle \text{segi-4 } ABCD = \sqrt{(s-a)(s-b)(s-c)(s-d)}.}$$

Contoh:



Diketahui:

$\triangle ABC$ dan perluasannya seperti gambar 17.

$\angle APB = \alpha$

$\angle BPQ = \beta$

$\angle PQB = \gamma$

Ditanya: Hitunglah PA, PB, QB, QC dan PQ.

Gambar 17.

Jawab:

Pada $\triangle BQC \longrightarrow$

$$\frac{BC}{\sin \delta} = \frac{BQ}{\sin y} \quad \text{atau } a : \sin \delta = BQ : \sin y$$

$$a \sin y = BQ \sin \quad .$$

Pada $\triangle PQB \longrightarrow$

$$\frac{BQ}{\sin \beta} = \frac{PB}{\sin \gamma} \quad \text{atau } BQ : \sin \beta = PB : \sin \gamma$$

$$BQ \sin \gamma = PB \sin \beta .$$

Pada $\triangle PAB \longrightarrow$

$$\frac{PB}{\sin x} = \frac{AB}{\sin \alpha} \quad \text{atau } PB : \sin x = c : \sin \alpha$$

$$PB \sin \alpha = c \sin x$$

$$PB = \frac{\sin x}{\sin \alpha}$$

$a \sin y = BQ \sin \delta$

$$a \sin y = \frac{PB \sin \beta}{\sin \gamma} \sin \delta$$

$a \sin y \sin \gamma = PB \sin \beta \sin \delta$

$$a \sin y \sin \gamma = \frac{c \sin x}{\sin \alpha} \sin \beta \sin \delta$$

$a \sin y \sin \gamma \sin \alpha = c \sin x \sin \beta \sin \delta$ atau

$a : \sin x \sin \beta \sin \delta = c$ z; $\sin y \sin \gamma \sin \alpha$

$\sin x : \sin y = a \sin \alpha \sin \gamma : c \sin \beta \sin \delta$.

Jika α, β, γ dan δ tertentu, maka harga x dan y dapat ditentukan.

Seterusnya didapat:

$$PA = \frac{c \sin (x + a)}{\sin \alpha} \qquad QC = \frac{a \sin (y + \delta)}{\sin \delta} \quad \text{dan}$$

$$PB = \frac{c \sin x}{\sin \alpha} \qquad PQ = \frac{PB \sin (\beta + \gamma)}{\sin \gamma}$$

$$QB = \frac{a \sin}{\sin \delta} \qquad = \frac{QB \sin (\beta + \gamma)}{\sin \beta}$$

<u>Soal-Soal</u>:

1. Dari $\triangle ABC$ diketahui $\angle B = 76^\circ$, AB = 18. Titik P dalam $\triangle ABC$, sehingga $\angle CPB = 39^\circ 43'$.
   Hitunglah PA, PB dan PC.

2. Dari $\triangle ABC$ diketahui, AB = 30, BC = 38 dan $\angle ABC = 66^\circ$.
   P dalam $\triangle ABC$, sehingga $\angle APB = 105^\circ$ dan $\angle BPC = 123^\circ$.
   Hitunglah PA, PB dan PC.

3. Dari segi-4 ABCD, $\angle A_1 = 29^\circ 25'$, $\angle A_2 = 41^\circ 37'$, $\angle C_1 = 27^\circ 2'$ dan $\angle C_2 = 46^\circ 21'$.
   Hitunglah sudut B_1, B_2, D_1 dan D_2.

4. Dari segi-4 ABCD, diketahui AB = 3, BC = 4, CD = 4,5, $\angle A = 78^\circ 20'$ dan $\angle D = 52^\circ 20'$. Hitunglah panjang AD.

5. Dari segi-4 ABCD diketahui b + c = a + d. Buktikanlah:

$$\frac{\cos^2 \frac{1}{2} A}{\cos^2 \frac{1}{2} C} = \frac{bc}{ad} \quad \text{dan} \quad \frac{\sin^2 \frac{1}{2} B}{\sin^2 \frac{1}{2} D} = \frac{cd}{ab}.$$

6. Dari segi-4 talibusur ABCD. a = 12,8, b = 15,3, c = 17,16 dan d = 20,27.

Hitunglah unsur-unsur yang lain.

7. Jika θ adalah sudut antara AB dan DC, buktikanlah pada segi-4 talibusur berlaku:

a). $\cos \frac{1}{2} \theta = (b + d)\sqrt{\frac{(s - b)(s - d)}{(ab -+ cd)(ad + bc)}}$

b). Tentukan pula $\sin \frac{1}{2} \theta$ dan $\operatorname{tg} \frac{1}{2} \theta$.

8. Buktikanlah pada segi-4 talibusur ABCD berlaku:

$$\sin = \frac{2\sqrt{(s - a)(s - b)(s - c)(s - d)}}{ac + bd}$$

$$\cos = \frac{b^2 + d^2 - a^2 - c^2}{2(ac + bd)}$$

9. Buktikanlah :

a). $\frac{\operatorname{tg} \frac{1}{2} A}{\operatorname{tg} \frac{1}{2} B} = \frac{s - d}{s - b}$

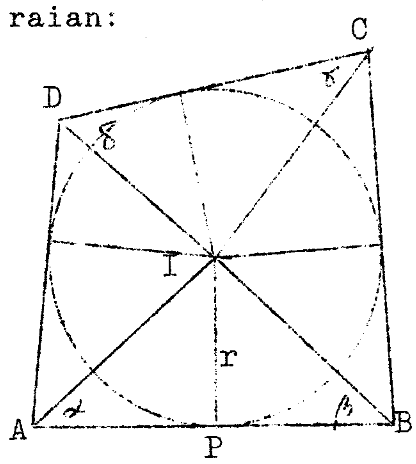
b). $\operatorname{tg} \frac{1}{2} A \operatorname{tg} \frac{1}{2} B = \frac{s - a}{s - c}$

10. Jika a + d = b + c, buktikanlah :

a). $\operatorname{tg} \frac{1}{2} A = \sqrt{\frac{ad}{bc}}$. b). $L = \sqrt{abcd}$.

SEGI EMPAT GARIS SINGGUNG

U raian:



Menurut Wydenes, (1953. hal.249).

AB = a

BC = b

CD = c

DA = d

Lingkaran (I,r) adalah lingkaran dalam segi-4 ABCD.

$$a + c = b + d$$

Gambar 18. Segi empat garis singgung.

Perhatikan $\triangle$AIB:

$$\text{cotg } \tfrac{1}{2} A = \frac{AP}{r} \longrightarrow AP = r \text{ cotg } \tfrac{1}{2} A$$

$$\text{cotg } \tfrac{1}{2} B = \frac{PB}{r} \longrightarrow PB = r \text{ cotg } \tfrac{1}{2} B$$

$$AP + PB = a$$

$$a = r \text{ cotg } \tfrac{1}{2} A + r \text{ cotg } \tfrac{1}{2} B$$

$$a = r(\text{cotg } \tfrac{1}{2} A + r \text{ cotg } \tfrac{1}{2} B)$$

$$r = \frac{a}{\text{cotg } \frac{1}{2} A + \text{cotg } \frac{1}{2} B}$$

$$r = \frac{a}{\dfrac{\text{cotg } \frac{1}{2} A}{\sin \frac{1}{2} A} + \dfrac{\text{cotg } \frac{1}{2} B}{\sin \frac{1}{2} B}}$$

$$\boxed{r = \frac{a \sin \frac{1}{2} A \quad \sin \frac{1}{2} B}{\sin \frac{1}{2} (A + B)}}$$

Menurut dalil sinus:

$$\frac{a}{\sin\{180 - (A + B)\}} = \frac{BI}{\sin \frac{1}{2} A}$$

$$\frac{a}{BI} = \frac{\sin(A + B)}{\sin \frac{1}{2} A}$$

$$\frac{BI}{CI} = \frac{\sin \frac{1}{2} C}{\sin \frac{1}{2} B} \quad \text{dan}$$

$$\frac{CI}{a} = \frac{\sin \frac{1}{2} D}{\sin \frac{1}{2} (C + D)}$$

Dengan menggunakan $\sin \frac{1}{2} (A + B) = \sin \frac{1}{2} (C + D)$

didapatkan:

$$\frac{a}{c} = \frac{a}{BI} \cdot \frac{BI}{CI} \cdot \frac{CI}{c}$$

$$= \frac{\sin \frac{1}{2} (A + B)}{\sin \frac{1}{2} A} \cdot \frac{\sin \frac{1}{2} C}{\sin \frac{1}{2} B} \cdot \frac{\sin \frac{1}{2} D}{\sin \frac{1}{2} (C + D)}$$

$$\boxed{\frac{a}{c} = \frac{\sin \frac{1}{2} C \ \sin \frac{1}{2} D}{\sin \frac{1}{2} A \ \sin \frac{1}{2} B}}$$

Dengan cara yang sama:

$$\frac{b}{a} = \frac{\sin \frac{1}{2} D \sin \frac{1}{2} A}{\sin \frac{1}{2} B \sin \frac{1}{2} C} \quad \text{dan}$$

$$\frac{ab}{cd} = \frac{\sin \frac{1}{2} C \sin \frac{1}{2} D}{\sin \frac{1}{2} A \sin \frac{1}{2} B} \cdot \frac{\sin \frac{1}{2} D \sin \frac{1}{2} A}{\sin \frac{1}{2} B \sin \frac{1}{2} C}$$

$$\boxed{\frac{ab}{cd} = \frac{\sin^2 \frac{1}{2} D}{\sin^2 \frac{1}{2} B}}$$

Perhatikan $\triangle$ ABC dan $\triangle$ ACD.

Menurut dalil cosinus:

$$\left.\begin{aligned} AC^2 &= a^2 + b^2 - 2ab \cos B \\ AC^2 &= c^2 + d^2 - 2cd \cos D \end{aligned}\right\} \quad (*)$$

Karena $a + c = b + d$

$a - b = d - c$

$a^2 - 2ab + b^2 = c^2 + d^2 - 2cd.$

Dari (*) didapat:

$$a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D$$
$$a^2 + 2ab - b^2 = c^2 + d^2 - 2cd. \quad -$$

$$-2ab\cos B + 2ab = -c\cos D + 2cd$$
$$2ab(1 - \cos B) = 2cd(1 - \cos D)$$

L (Luas) segi empat ABCD = $\frac{1}{2}$ ab sin B + $\frac{1}{2}$ cd sin D

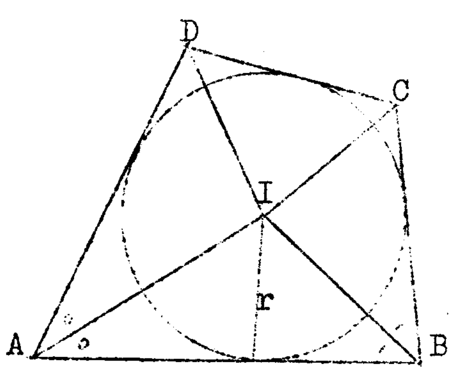
$$\frac{ab}{\sin^2 \frac{1}{2} D} = \frac{cd}{\sin^2 \frac{1}{2} B} \qquad \text{umpama} = k$$

$$ab = k\sin^2 \tfrac{1}{2} D \quad \text{dan}$$
$$cd = k\sin^2 \tfrac{1}{2} B$$
$$L = \tfrac{1}{2} k(\sin B\sin^2 \tfrac{1}{2} D + \sin D\sin^2 \tfrac{1}{2} B)$$
$$= k\sin \tfrac{1}{2} B\sin \tfrac{1}{2} D\sin \tfrac{1}{2}(B + D)$$
$$k^2\sin^2 \tfrac{1}{2} B\sin^2 \tfrac{1}{2} D = abcd \quad \text{dan}$$
$$\sin \tfrac{1}{2}(A + C) = \sin \tfrac{1}{2}(B + D)$$

maka:

$$\boxed{L \text{ segi-4 garis singgung } ABCD = \sqrt{abcd}.\sin \tfrac{1}{2}(A + C)}$$

Contoh:



Gambar 19.

Dari segi-4 garis singgung ABCD (gambar 19).

AB = a, BC = b, CD = c dan DA = d.

r = jari-jari lingkaran dalam segi-4 ABCD.

$a + c = b + d$

$a \geq b$ dan $c \leq d$

$a \geq d$ dan $b \geq c$

sehingga $a \geq b \geq d \geq c$

$$\angle A = 2\alpha$$
$$\angle B = 2\beta$$
$$\angle C = 2\gamma$$
$$\angle D = 2\delta$$
$$a = r \operatorname{cotg}\alpha + r \operatorname{cotg}\beta .$$
$$b = r \operatorname{cotg}\beta + r \operatorname{cotg}\gamma .$$
$$c = r \operatorname{cotg}\gamma + r \operatorname{cotg}\delta .$$
$$d = r \operatorname{cotg}\delta + r \operatorname{cotg}\alpha .$$
$$\alpha + \beta + \gamma \quad \delta = 180^\circ .$$

$$\operatorname{cotg}(\alpha + \beta) + \operatorname{cotg}(\gamma + \delta) = 0$$

$$\frac{\operatorname{cotg}\alpha \operatorname{cotg}\beta - 1}{\operatorname{cotg}\alpha + \operatorname{cotg}\beta} + \frac{\operatorname{cotg}\gamma \operatorname{cotg}\delta - 1}{\operatorname{cotg}\gamma + \operatorname{cotg}\delta} = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\operatorname{cotg}\alpha \operatorname{cotg}\beta - 1}{a} + \frac{\operatorname{cotg}\gamma \operatorname{cotg}\delta - 1}{b} = 0$$

umpama: $r \operatorname{cotg}\alpha = x$ dan $\operatorname{cotg}\beta = \dfrac{ax}{r}$

$\operatorname{cotg}\gamma = \dfrac{x - (a - b)}{r}$ dan $\operatorname{cotg}\delta = \dfrac{d - x}{r}$

Semua ini disubstitusikan ke persamaan (1).

Didapat:

$$c\{x(a - x) - r^2\} + a\{(b - a + x)(d - x) = r^2\} = 0$$
$$(a + c)x^2 + (ab - rc - a^2 - ad)x + (a + c)r^2 + ad(a - b) = 0$$
$$D = b^2 - ac \geq 0$$

$$x = \frac{ad \pm \sqrt{abcd - (a + c)^2 r^2}}{a + c}$$

$$= \frac{ad}{a + c} \pm \sqrt{\frac{abcd}{(a + c)^2} - r^2}$$

Soal-Soal

Buktikanlah pada segi-4 garis singgung ABCD berlaku:

1. pq os $\varphi = ac - bd$.

2. $a \sin^2 \frac{1}{2} A + c \sin^2 \frac{1}{2} D = b \sin^2 \frac{1}{2} (A + B)$.

3. $L = \frac{1}{2} (ac - bd) \operatorname{tg} \varphi$.

4. $4L^2 = p^2q^2 - (ac - bd)^2$.

5. a). Buktikanlah dalam segi-4 ABCD berlaku:

$$\sin(\alpha + \tfrac{1}{2}\beta) = \frac{a + c}{b}\sqrt{\frac{(s - a)(s - c)}{ac}}$$

Hitunglah Untuk $\cos(\alpha + \frac{1}{2}\beta)$

GRAFIK FUNGSI TRIGONOMETRI
(lanjutan)

Syarat yang perlu. dan cukup nuntuk menggambarkan grafik fungsi trigonometri adalah:

a. Sederhanakan fungsi itu,

b. Tentukan harga ekstrim,

c. Tentukan titik potong dengan kedua sumbu,

d. Tentukan titik-titik lain yang memenuhi fungsi.

Kemudian baru digambarkan grafik selengkapnya.

Contoh:

1. Gambarkanlah grafik $y = \dfrac{5\cos x + 3}{4\cos x - 2}$

   dalam interval $0^\circ \leq x \leq 360^\circ$.

   Jawab:

$$y = \frac{5\cos x + 3}{4\cos x - 2}$$

$$= \frac{5}{4} + \frac{1\frac{1}{2}}{4\cos x - 2}$$

untuk $x = 0$ dan $x = 360^\circ$
$y_{max} = 4$.

$$y_{ril\ max} = \frac{5\cos 180^\circ + 3}{4\cos 180^\circ - 2} = \frac{1}{3}$$

untuk $x = 180^\circ$.

Asimtot $y = \infty \longrightarrow 4\cos x - 2 = 0$

$$4\cos x = 2$$
$$\cos x = \tfrac{1}{2}$$
$$x_1 = 60^\circ$$
$$x_2 = 300^\circ.$$

Titik potong dengan sumbu X $\longrightarrow y = 0$

$$y = \frac{5\cos x + 3}{4\cos x - 2}$$

$$0 = \frac{5\cos x + 3}{4\cos x - 2}$$

$$\cos x = -\tfrac{3}{5}$$

$x_1 = 126^o52'13"$
$x_2 = 233^o7'47"$

Titik potong sumbu Y $\longrightarrow x = 0$

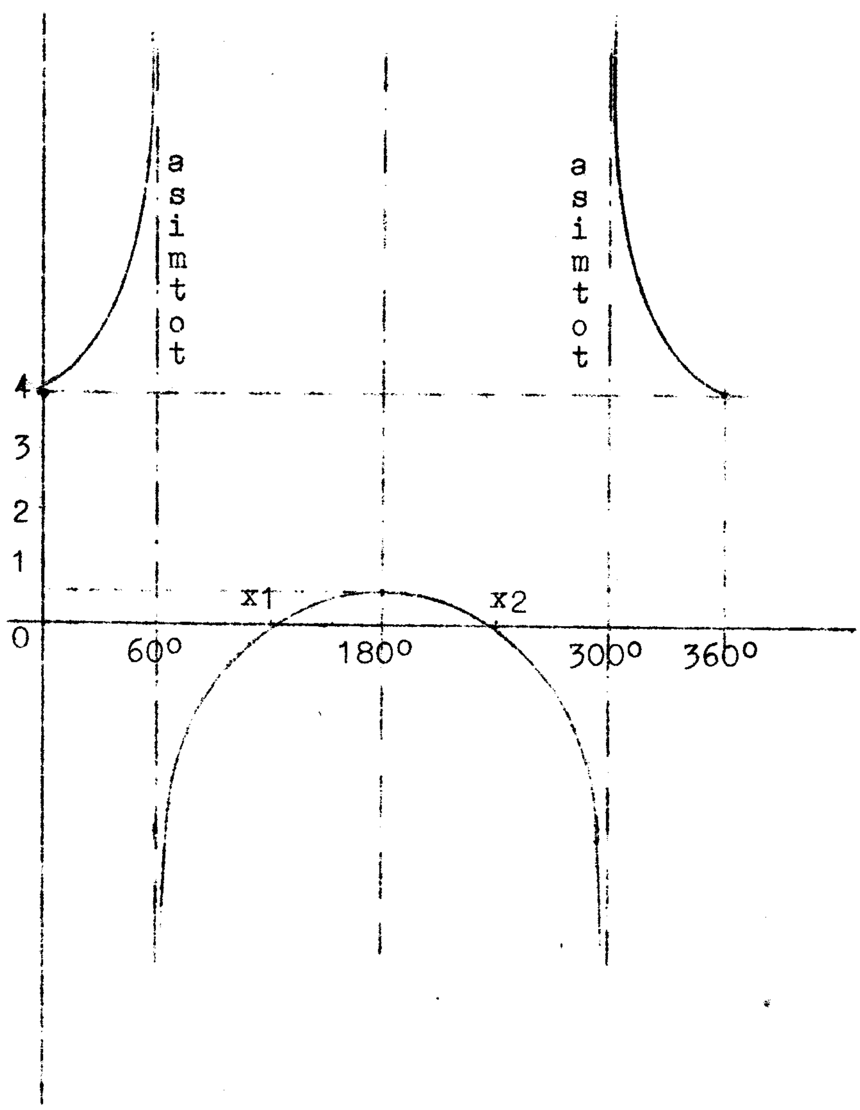
$$\left.\begin{array}{l} x = 0 \\ y = \dfrac{5\cos x + 3}{4\cos x - 2} \end{array}\right\} \quad (0,4)$$

Titik-titik lain yang memenuhi untuk:

$$x = 30^o \text{ dan } 330^o \longrightarrow y = \frac{5\cos 30^o + 3}{4\cos 30^o - 2} = 5,00652$$

$$\text{untuk } x = 45^o \text{ dan } 315^o \longrightarrow y = \frac{5\cos 45^o + 3}{4\cos 45^o - 2} = 8,125$$

$$\text{Grafik} \quad y = \frac{5\cos x + 3}{4\cos x + 2}$$



Gambar 19.

2. Gambarlah grafik $y = \dfrac{4\ tg^2 x + 5\ tg\ x - 10}{tg^2 x + tg\ x - 2}$

dalam interval $0^o \le x \le 180^o$.

Jawab: Titik potong sumbu X ——> y = 0

$$4\ tg^2 x + 5\ tg\ x - 10 = 0$$

$$tg\ x_1 = \frac{-5 \pm \sqrt{25 + 160}}{8}$$

$$= \frac{-5 \pm \sqrt{185}}{8}$$

$$tg\ x_1 = \frac{-5 + \sqrt{185}}{8} = 1{,}07518 \longrightarrow x_1 = 47^o4'30''$$

$$tg\ x_2 = \frac{-5 - \sqrt{185}}{8} = -2{,}32518 \longrightarrow x_2 = 113^o16'17''$$

Syarat asimtot tegak y = ——> $tg^2 x + tg\ x - 2 = 0$.

$$tg\ x_{1.2} = \frac{-1 \pm \sqrt{1 + 8}}{2} = \frac{-1 \pm 3}{2}$$

$$tg\ x_1 = \frac{-1 + 3}{2} = 1 \longrightarrow x_3 = 45^o$$

$$tg\ x_2 = \frac{-1 - 3}{2} = -2 \longrightarrow x_4 = 116^o33'54''.$$

Titik potong sumbu Y ——> x = 0

$$y = \frac{4.0 + 5.0 - 10}{0 + 0 - 2} = 5 \qquad (tg\ 0 = 0)$$

y_{Ext} didapat dari:

$$(y - 4)tg^2 x + (y - 5)\ tg\ x - (2y - 10) = 0$$

$$D \ge 0$$

$$(y - 5)^2 + 8(y - 4)(y - 5) \ge 0$$

$$(y - 5)(9y - 37) \ge 0$$

$$y \le 4\tfrac{1}{9}$$

$$y \ge 5.$$

y_{max} relatif $= 4\frac{1}{9}$

y_{min} relatif $= 5$.

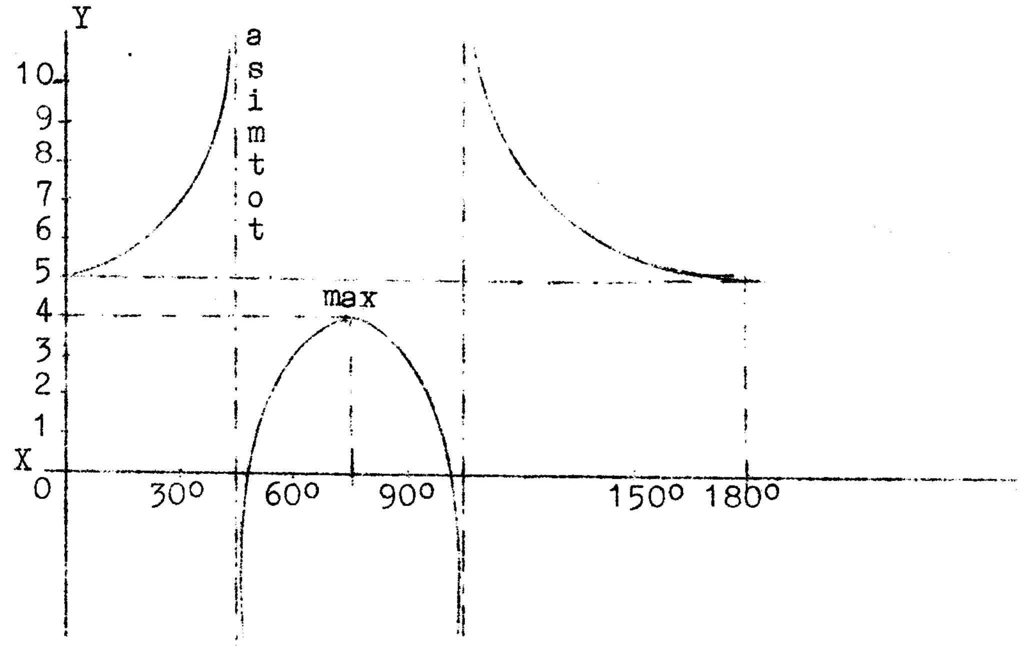
Untuk $y = 4\frac{1}{9}$ ——> $tg\ x = 4$

$$x = 75^o57'50''$$

$y = 5$ ——> $tg\ x = 0$

$x = 0^o$ dan $x = 180^o$

Grafik $y = \dfrac{4\ tg^2x + 5\ tg\ x - 10}{tg^2x + tg\ x - 2}$



Gambar 20.

Soal-Soal:

Gambarkanlah grafik dibawah ini dalam interval $0^o \leq x \leq 360^o$.

1. $y = 2 \sin^2x + 5 \sin x - 3$.

2. $y = -2 \cos^2x + 3 \cos x - 2$.

3. $y = \sec^2x + 4 \sec x + 5$.

4. $y = \dfrac{8 \sin x + 3}{2 \sin x - 1}$.

5. $y = \dfrac{(1 + \cos x)^2}{\cos x(1 - \cos x)}$

6. $y = \dfrac{\sin x}{3 + \cos x}$.

Gambarkanlah grafik dibawah ini dalam interval $0^\circ \leq x \leq 180^\circ$.

7. $y = \frac{1}{5} \operatorname{tg}^2 x - \frac{1}{2} \operatorname{tg} x - 1.$

8. $\dfrac{28 \sin 2x - 16 \cos 2x - 16 \cos^2 x}{6 \cos 2x - \sin 2x + 8}.$

9. $y = 4 \operatorname{tg} x + 9 \operatorname{cotg} x - 3.$

10. $y = \dfrac{49 - 10 \sin 2x - 45 \sin^2 x}{2 \sin 2x - 12 \cos^2 x}.$

FUNGSI CYCLOMETRY

(lanjutan)

Uraian:

Selisih dan jumlah dari arc dengan buktinya.

1. arc sin p - arc sin q = arc sin x ?

Umpama: $\alpha = \text{arc sin } p \longrightarrow \sin\alpha = p \;,\; \cos \quad = \sqrt{1 - p^2}$

$\beta = \text{arc sin } q \longrightarrow \sin\beta = q \;,\; \cos \quad = \sqrt{1 - q^2}$

$\alpha - \beta = \text{arc sin } x \longrightarrow x = \sin(\alpha - \beta)$

$= \sin\alpha \cos\beta - \cos\alpha \sin\beta$

$= \sqrt{1 - q^2} - \sqrt{1 - p^2}.\ q$

$= p\sqrt{1 - q^2} - q\sqrt{1 - p^2}.$

$$\boxed{\text{arc sin } p - \text{arc sin } q = \text{arc sin}(p\sqrt{1 - q^2} - q\sqrt{1 - p^2}).}$$

2. arc cos p + arc cos q = arc cos x.

Umpama: $\alpha = \text{arc cos } p \longrightarrow \cos\alpha = p \;,\; \sin\alpha = \sqrt{1 - p^2}$

$\beta = \text{arc cos } q \longrightarrow \cos\beta = q \;,\; \sin\beta = \sqrt{1 - q^2}$

$\alpha + \beta = \text{arc cos } x \longrightarrow x = \sin(\alpha + \beta)$

$= \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$= pq - \sqrt{1 - p^2} . \sqrt{1 - q^2}$

$= pq - \sqrt{(1 - p^2)(1 - q^2)}$

$$\boxed{\text{arc cos } p + \text{arc cos } q = \text{arc cos}[pq - \sqrt{(1 - p^2)(1 - q^2)}]}$$

3. $\text{arc sin } p + \text{arc sin } q = \text{arc cos}\sqrt{1 - p^2} + \text{arc cos}\sqrt{1 - q^2}$

$= \text{arc cos}\{\sqrt{(1 - p^2)(1 - q^2)} - pq\}$

$= \frac{1}{2}\pi - \text{arc sin}\{\sqrt{(1-p^2)(1-q^2)} - pq\}$

$= \frac{1}{2}\pi + \text{arc sin}\{pq - \sqrt{(1-p^2)(1-q^2)})$

$$\therefore \boxed{\text{arc sin } p + \text{arc sin } q = \text{arc sin}\{pq - \sqrt{(1-p^2)(1-q^2)}) + \tfrac{1}{2}\pi.}$$

4. $\text{arc cos } p - \text{arc cos } q = \text{arc sin}\sqrt{1 - p^2} - \text{arc sin}\sqrt{1 - q^2}$

$$= \text{arc sin}\{q\sqrt{1 - p^2} - p\sqrt{1 - q^2}\}$$

$$= \tfrac{1}{2}\pi - \text{arc cos}(q\sqrt{1-p^2} - p\sqrt{1-q^2})]$$

$$= \tfrac{1}{2}\pi - [\pi - \text{arc cos}\{p\sqrt{1-q^2} - q\sqrt{1-p^2}\}]$$

$$= \text{arc cos}[p\sqrt{1-q^2} - q\sqrt{1-p^2} - \tfrac{1}{2}\pi].$$

$$\therefore \boxed{\text{arc cos } p - \text{arc cos } q = \text{arc cos }[p\sqrt{1-q^2} - q\sqrt{1-p^2}] - \tfrac{1}{2}\pi}$$

5. Menurut ***Wydenes***, (1953, hal. 289);
dengan meumpamakan arc tg p $=\alpha$ dan arc tg q $=\beta$, maka di dapat:

$$\text{arc tg } p - \text{arc tg } q = \text{arc tg } \frac{p - q}{pq + 1}$$

dan dengan meumpamakan arc cotg p $=\alpha$ dan arc cotg q $=\beta$ maka didapat pula:

$$\text{arc cotg } p + \text{arc cotg } q = \text{arc cotg } \frac{pq - 1}{p + q} .$$

6. $\text{arc tg } p + \text{arc tg } q = \text{arc cotg } \frac{1}{p} + \text{arc cotg } \frac{1}{q}$

$$= \text{arc cotg } \frac{\frac{1}{pq} - 1}{\frac{1}{p} + \frac{1}{q}} = \quad \text{(menurut nomor 5)}$$

$$= \text{arc cotg } \frac{1 - pq}{p + q}$$

$$= \tfrac{1}{2}\pi - \text{arc tg } \frac{1 - pq}{p + q}$$

$$= \tfrac{1}{2}\pi + \text{arc tg } \frac{pq - 1}{p + q}$$

$$\therefore \boxed{\text{arc tg } p + \text{arc tg } q = \text{arc tg } \frac{pq - 1}{p + q} + \tfrac{1}{2}\pi .}$$

7. $\text{arc cotg } p - \text{arc cotg } q = \text{arc tg } \frac{1}{p} - \text{arc tg } \frac{1}{q}$

$$= \text{arc tg } \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{pq} + 1} = \text{(menurut nomor 5)}$$

$$= \text{arc tg } \frac{q - p}{1 + pq}$$

$$= \frac{1}{2}\pi - \text{arc cotg } \frac{q - p}{1 + pq}$$

$$= \frac{1}{2}\pi - \left(\pi - \text{arc cotg } \frac{p + q}{1 + pq}\right)$$

$$= \text{arc cotg } \frac{p + q}{1 + pq} - \frac{1}{2}\pi$$

$$\therefore \boxed{\text{arc cotg } p - \text{arc cotg } q = \text{arc cotg } \frac{p - q}{1 + pq} - \frac{1}{2}\pi}$$

Penyelesaian Soal-soal Campuran

1. Buktikanlah:

$$R = \frac{s}{\sin\alpha + \sin\beta + \sin\gamma}$$

$$= \frac{\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c}{\sin\alpha + \sin\beta + \sin\gamma}$$

$$= \frac{R\sin\alpha + R\sin\beta + R\sin\gamma}{\sin\alpha + \sin\beta + \sin\gamma}$$

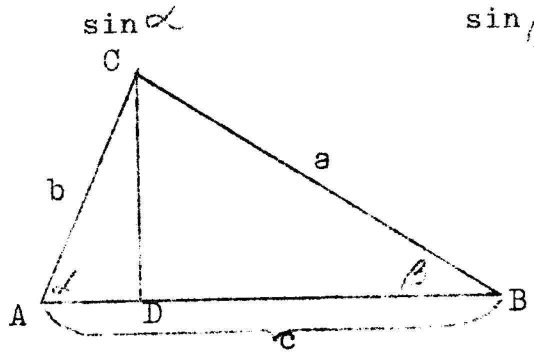
$$= \frac{R(\sin\alpha + \sin\beta + \sin\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}$$

$R = R \longrightarrow$ terbukti.

2. Buktikanlah:

$$\sum \frac{b\cos\gamma + c\cos}{\sin\alpha} = 6R$$

$$\frac{b\cos\gamma + c\cos\beta}{\sin\alpha} + \frac{c\cos\alpha + a\cos\gamma}{\sin\beta} + \frac{a\cos\beta + b\cos\alpha}{\sin\gamma} =$$



$$\begin{aligned} AD &= b\cos\alpha \\ BD &= a\cos\beta \\ \hline c &= b\cos\alpha + a\cos\beta \\ c &= a\cos\beta + b\cos\alpha \end{aligned}$$

maka persamaan di atas menjadi:

$$\frac{a}{\sin\alpha} + \frac{b}{\sin\beta} + \frac{c}{\sin\gamma} =$$

$2R + 2R + 2R = 6R.$

3. Buktikanlah arc tg 2 + arc tg 3 = $\frac{3}{4}\pi$

Bukti:

arc tg 2 + arc tg 3 =

$$\text{arc tg } \frac{6 - 1}{5} + \frac{1}{2}\pi =$$

$$\text{arc tg } 1 + \frac{1}{2}\pi =$$

$$\frac{1}{4}\pi + \frac{1}{2}\pi =$$

$$\frac{3}{4}\pi = \frac{3}{4}\pi.$$

4. Buktikanlah arc tg $\frac{1}{3}$ - arc tg $\frac{1}{4}$ = arc tg $\frac{1}{13}$

Bukti:

arc tg $\frac{1}{3}$ - arc tg $\frac{1}{4}$ =

$$\text{arc tg } \frac{\frac{1}{3} - \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{4} + 1} =$$

$$\text{arc tg } \frac{\frac{1}{12}}{\frac{1}{12} + 1} =$$

$$\text{arc tg } \frac{1}{13} = \text{arc tg } \frac{1}{13}.$$

5. Buktikanlah:

$$1A = \frac{s - a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c - a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{\frac{1}{2}b + \frac{1}{2}b - \frac{1}{2}a}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R \sin\beta + R \sin\gamma - R \sin\alpha}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R[2 \sin \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2} - \sin(\beta + \gamma)]}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R[2 \sin \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2} - 2 \sin \frac{\beta+\gamma}{2} \cos \frac{\beta+\gamma}{2}]}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R[2 \sin \frac{\beta+\gamma}{2} (\cos \frac{\beta-\gamma}{2} - \cos \frac{\beta+\gamma}{2})]}{\cos \frac{1}{2}\alpha}$$

$$= \frac{R\ 2 \cancel{\cos \frac{1}{2}\alpha} - 2 \sin \frac{\beta-\gamma+\beta+\gamma}{4} \sin \frac{\beta-\gamma-\beta-\gamma}{4}}{\cancel{\cos \frac{1}{2}\alpha}}$$

$$= -4R \sin \tfrac{1}{2}\beta \sin (\tfrac{-1}{2} \gamma)$$

$$1A = 4R \sin \tfrac{1}{2} \beta \sin \tfrac{1}{2} \gamma$$

$1A = 1A$ ———> terbukti.

6. Buktikanlah: $a^2 + HA^2 = b^2 + HB^2 = c^2 + HC^2 = 4R^2$.

Bukti:

$$\begin{aligned} a^2 + HA^2 &= (2R \sin \alpha)^2 + (2R \cos \alpha)^2 \\ &= 4R^2 \sin^2\alpha + 4R^2 \cos^2\alpha \\ &= 4R^2(\sin^2\alpha + \cos^2\alpha) \\ &= 4R^2. \end{aligned}$$

$$\begin{aligned} b^2 + HA^2 &= (2R \sin \beta)^2 + (2R \cos \beta)^2 \\ &= 4R^2 \sin^2\beta + 4R^2 \cos^2\beta \\ &= 4R^2(\sin^2\beta + \cos^2\beta) \\ &= 4R^2. \end{aligned}$$

$$\begin{aligned} c^2 + HA^2 &= (2R \sin \gamma)^2 + (2R \cos \gamma)^2 \\ &= 4R^2 \sin^2\gamma + 4R^2 \cos^2\gamma \\ &= 4R^2(\sin^2\gamma + \cos^2\gamma) \\ &= 4R^2. \end{aligned}$$

$\therefore\ a^2 + HA^2 = b^2 + HB^2 = c^2 + HC^2 = 4R^2$.

7. Buktikanlah dalam $\triangle ABC$, berlaku:

$$\sum \frac{a^2 \sin(\beta-\gamma)}{\sin \beta + \sin \gamma} = 0$$

$$\frac{a^2\sin(\beta-\gamma)}{\sin\beta+\sin\gamma}+\frac{b^2\sin(\gamma-\alpha)}{\sin\gamma+\sin\alpha}+\frac{c^2\sin(\alpha-\beta)}{\sin\alpha+\sin\beta}=$$

$$4R^2\left[\frac{\sin^2\alpha\sin(\beta-\gamma)}{\sin\beta+\sin\gamma}+\frac{\sin^2\beta\sin(\gamma-\alpha)}{\sin\gamma+\sin\alpha}+\frac{\sin^2\gamma\sin(\alpha-\beta)}{\sin\alpha+\sin\beta}\right]=$$

$$4R^2\left[\frac{\sin^2\alpha\,.\,\cancel{2}\sin(\frac{\beta-\gamma}{2})\cos(\frac{\beta-\gamma}{2})}{\cancel{2}\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}}+\frac{\sin^2\beta\,2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2}}{2\sin\frac{\gamma+\alpha}{2}\cos\frac{\gamma-\alpha}{2}}+\right.$$

$$\left.\frac{\sin^2\gamma\,2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}\right]=$$

$$4R^2\left[\frac{\sin^2\alpha\sin(\frac{\beta-\gamma}{2})}{\sin\frac{\beta+\gamma}{2}}+\frac{\sin^2\beta\sin\frac{\gamma-\alpha}{2}}{\sin\frac{\gamma+\alpha}{2}}+\frac{\sin^2\gamma\sin\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}}\right]=$$

$$4R^2\left[\frac{\sin^{\cancel{2}}\alpha\,.\,\sin(\beta+\gamma)\sin\frac{\beta-\gamma}{2}}{\sin\frac{\beta+\gamma}{2}}+\frac{\sin\beta\sin(\gamma+\alpha)\sin\frac{\gamma-\alpha}{2}}{\sin\frac{\gamma+\alpha}{2}}+\right.$$

$$\left.\frac{\sin\gamma\sin(\alpha+\beta)\sin\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}}\right]=$$

$$4R^2[2\sin\alpha\cos\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}+2\sin\beta\cos\frac{\gamma+\alpha}{2}\sin\frac{\quad}{2}+$$
$$2\sin\gamma\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}]=$$

$$4r^2[\sin\alpha(\sin\beta-\sin\gamma)+\sin\beta(\sin\gamma-\sin\alpha)+\sin\gamma(\sin\alpha-\sin\beta)]=$$

$$4R^2[\sin\alpha\sin\beta-\sin\alpha\sin\gamma+\sin\beta\sin\gamma-\sin\beta\sin\alpha+\sin\gamma\sin\alpha-\sin\gamma\sin\beta]=$$

$4R^2[\ 0\ ] = 0.$ ——> terbukti.

8. Dari $\triangle ABC$, diketahui $\sin^2\gamma = \sin 2\alpha\,\sin 2\beta$.

Selidikilah bentuk $\triangle$ itu.

Jawab:

$\sin^2\gamma = \sin 2\alpha \sin 2\beta$

$2\sin^2\gamma = 2\sin 2\alpha \sin 2\beta$

$2\sin^2\gamma = \cos(2\alpha - 2\beta) - \cos(2\alpha + 2\beta)$

$1 - \cos 2\gamma = \cos(2\alpha - 2\beta) - \cos(2\alpha + 2\beta)$

$1 - \cos(2\alpha + 2\beta) = \cos(2\alpha - 2\beta) - \cos(2\alpha + 2\beta)$

$1 = \cos(2\alpha - 2\beta)$

$\cos(2\alpha - 2\beta) = \cos 0$

$$2\alpha - 2\beta = 0$$
$$2\alpha = 2\beta$$
$$\alpha = \beta$$

$\therefore$ $\triangle ABC$ sama kaki.

9. Pada $\triangle ABC$, diketahui $a + c = 2b$.

   Buktikanlah:

   a). $\cos \frac{1}{2} (\alpha - \gamma) = 2 \sin \frac{1}{2}\beta$

   b). $\operatorname{tg} \frac{1}{2}\alpha \operatorname{tg} \frac{1}{2}\gamma = \frac{1}{3}$

   Bukti:

   a. $a + c = 2b$

$$2R \sin\alpha + 2R \sin\gamma = 4R \sin\beta$$
$$\sin\alpha + \sin\gamma = 2 \sin\beta$$
$$2 \sin\frac{\alpha+\gamma}{2} \cos\frac{\alpha-\gamma}{2} = 4 \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta$$
$$2 \cos \frac{1}{2}\beta \cos\frac{\alpha-\gamma}{2} = 4 \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta$$
$$\cos\frac{\alpha-\gamma}{2} = \frac{2 \sin \frac{1}{2}\beta \cancel{\cos \frac{1}{2}\beta}}{\cancel{\cos \frac{1}{2}\beta}}$$
$$\cos \frac{1}{2} (\alpha - \gamma) = 2 \sin \frac{1}{2}\beta \longrightarrow \text{terbukti.}$$

   b. $\operatorname{tg} \frac{1}{2}\alpha \operatorname{tg} \frac{1}{2}\gamma = \frac{1}{3}$

$$\cos(\tfrac{1}{2}\alpha - \tfrac{1}{2}\gamma) \; 2 \sin \tfrac{1}{2}\beta$$
$$\cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\gamma + \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\gamma = 2 \cos(\tfrac{1}{2}\alpha + \tfrac{1}{2}\gamma)$$
$$\cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\gamma + \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\gamma =$$
$$2 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\gamma - 2 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\gamma$$
$$3 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\gamma = \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\gamma$$
$$\frac{3 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\gamma}{\cos \frac{1}{2}\alpha \cos \frac{1}{2}\gamma} = 1$$
$$3 \operatorname{tg} \tfrac{1}{2}\alpha \operatorname{tg} \tfrac{1}{2}\gamma = 1$$
$$\operatorname{tg} \tfrac{1}{2}\alpha \operatorname{tg} \tfrac{1}{2}\gamma = \tfrac{1}{3}. \longrightarrow \text{terbukti.}$$

10. Dari $\triangle ABC$ ($\gamma = 90^{o}$) diketahui: $a + b = p$ dan $b + c = q$. Hitunglah unsur-unsur $\triangle$ itu dalam p dan q.

Jawab:

$a + b = p \longrightarrow a = p - b$

$b + c = q \longrightarrow c = q - b$

Menurut dalil *Pytagoras*:

$a^2 + b^2 = c^2$

$(p - b)^2 + b^2 = (q - b)^2$

$p^2 - 2bp + b^2 + b^2 = q^2 - 2qb + b^2$

$b^2 - 2(p - q)b + (p^2 - q^2) = 0$

$D = 8q(q - p) \geq 0 \longrightarrow p \leq q.$

$$b = \frac{2(p - q) \pm \sqrt{8q(q - p)}}{2}$$

$$b = \frac{2(p - q) \pm 2\sqrt{2q(q - p)}}{2}$$

$b_1 = p + q\ \sqrt{2q(q - p)}$

$b_2 = p - q\ \sqrt{2q(q - p)} \longrightarrow$ tidak memenuhi.

$a = p - b$

$a = p - (p - q + \sqrt{2q(q - p)}$

$a = q - \sqrt{2q(q - p)}.$

$a = q - b$

$a = q - (p - q + \sqrt{2q(q - p)}$

$c = 2q - p - \sqrt{2q(q - p)}.$

Soal-Soal Tambahan:

1. Buktikanlah:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2(\alpha+\beta+\gamma) = 2 + 2\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$$

2. Hitunglah x dari persamaan:

   a). $12\sin^2 x - \text{tg } x - 4 \text{ tg } x = 1.$

   b). $\cos 4x + \cos 2x + \cos x = 0.$

   c). $\cos^3 x - \sin x \cos x - \sin^3 x = 1.$

3. Jika $\text{tg}^2\alpha = 1 + 2\text{ tg}^2\beta$; buktikanlah :

   $\cos 2\beta = 1 + 2\cos 2\alpha$.

4. Dari $\triangle ABC$, diketahui $\text{tg }\alpha + \text{tg }\gamma = \text{tg }\beta$.

   Buktikanlah:

   a). $\text{tg }\alpha \text{ tg }\gamma = 2.$

   b). Luas $\triangle ABC = 2R^2 \sin 2\beta$.

5. Gambarlah grafik fungsi terigonometri :

   $y = 12\sin^2 x - 8\sin x + 1$ dalam interval $0^\circ \leq x \leq 360^\circ$.

6. Hitunglah unsur-unsur yang lain dari segi-4 garis singgung, jika diketahui $a = 23{,}2$, $c = 36{,}8$, $\angle A = 82^\circ 40'$ dan $\angle B = 95^\circ 20'$.

7. Hitunglah x dan y dari persamaan:

   $\cos x + \cos y = 1$

   $\text{tg } x + \text{tg } y = 2(\sin x + \sin y).$

8. Hitunglah x dari:

$$x = \left(\frac{\sin 1^\circ 13' \text{ tg } 0^\circ 48'}{\cos 89^\circ 14' \text{ cotg } 88^\circ 36'}\right)^{10}$$

9. Buktikanlah: $\text{cotg } 6^\circ \text{ tg } 12^\circ \text{ tg } 24^\circ \text{ tg } 48' = 1.$

10. α, β dan γ adalah sudut-sudut dari $\triangle ABC$, buktikanlah:

$\sum \operatorname{tg}(45^\circ - \frac{1}{4}\alpha) \operatorname{tg}(45^\circ - \frac{1}{4}\beta) = 1.$

11. Jika $\alpha + \beta + \gamma = 180^\circ$, buktikanlah:

$\sum \operatorname{tg}\alpha \operatorname{cotg}\beta \operatorname{cotg}\gamma = \sum \operatorname{tg}\alpha - 2\sum \operatorname{cotg}\alpha$.

12. Buktikanlah:

$\sum \sin 2\alpha \cos\beta \cos\gamma \sin(\beta - \gamma) = 0.$

13. Buktikanlah:

$\sum \cos 2\alpha \cos\beta \cos\gamma \sin(\beta - \gamma) = \sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \beta).$

14. Hitunglah x dari:

$\operatorname{arc\,tg} \frac{1}{2} + \operatorname{arc\,tg} \frac{1}{4} - \operatorname{arc\,tg} \frac{11}{27} = \operatorname{arc\,tg} x.$

15. Jika $x + y + z = 90^\circ$, buktikanlah:

$\cos x + \cos y + \cos z = 4\cos(45^\circ - \frac{1}{2}x)\cos(45^\circ - \frac{1}{2}y)\cos(45^\circ - \frac{1}{2}z).$

16. Dari segitiga ABC, diketahui:

$a^2\sin^2\beta + b^2\sin^2\alpha = 2ab\cos\alpha\cos\beta$.

Selidikilah bentuk segitiga itu.

17. Hitunglah x dari:

$$\operatorname{tg} \frac{1}{3}x - 14^\circ) = \frac{\operatorname{tg}^4 153^\circ 21'}{\sin^3 213^\circ 20' \cos^5 266^\circ 13'}.$$

18. Hitunglah x dan y dari:

a). $\begin{cases} \operatorname{tg} x = \sin 3y \\ \sin x = \operatorname{tg} 3y. \end{cases}$

b). $\begin{cases} \operatorname{tg} x = \cos 3y \\ \cos x = \operatorname{tg} 3y. \end{cases}$

19. Hitunglah jumlah dari :

$\sin a \sin 3a + \sin 2a \sin 6a + \sin 4a \sin 12a + \ldots$

20. Buktikanlah dalam segitiga berlaku:

$\sum (a^2 - b^2)\operatorname{cotg} \gamma = 0.$

21. Dari $\triangle ABC$ ($\gamma = 90^\circ$), diketahui:

$a + b = 97{,}34$ dan $b + c = 128{,}87$.

Hitunglah unsur-unsur $\triangle$ itu.

22. Dari $\triangle ABC$ ($\gamma = 90^\circ$), diketahui $a - b = v$ dan $\alpha - \beta = \varphi$. Hitunglah unsur-unsur $\triangle$ itu dalam v dan φ.

23. Eliminirlah φ dari:

$\sin \varphi + \cos \varphi = a$

$\sin^3 \varphi + \cos^3 \varphi = b.$

24. Gambarlah grafik fungsi:

$y = \dfrac{5 - 3 \cos x}{\sin x}$ dalam interval $0^\circ \leq x \leq 360^\circ$.

25. Hitunglah x dari: $15 \operatorname{tg}^2 x - \operatorname{tg} x \sec^2 x = 2 \sec^4 x$.

26. Dari $\triangle ABC$, diketahui $a = 834{,}75$; $b = 698{,}23$ dan $d_\gamma = 725{,}83$. Hitunglah unsur-unsur yang lain.

DAFTAR PUSTAKA

1. Alders, C.J, __Ilmu Ukur Segi Tiga__, Pradnya Paramita, Jalan Medium 8 - Jakarta 1969.

2. Christian. R.Robert: __A.Brief Trigonometry__, Waltham Massuchusetts. Toronto - London.

3. J. Pignany, Tullio and Haggard Paul, __Element of Trigonometry__, Harcourt, Brace & World, Inc.1968.

4. Kobus M.L. Van Thijn, A Dr. dan Rawuh Rd, __Ilmu Ukur Segi Tiga__, J.B, Wolters, Jakarta Groningen, 1953.

5. Marcus, Marvin and Mine Hendryk, __College Trigonometry__, Hongliton Mifflin Company, Boston, 1971.

6. Wijdenes, __Goniometri en Trigonometry__, P. Noordholf NV, Groningen, Jakarta, 1953.

7. Universitas Terbuka, __Trigonometri__.